Hawking radiation in the swiss cheese universe

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Abstract

The Hawking radiation forms the essential basis of the black hole thermodynamics. The black hole thermodynamics denotes a nice correspondence between black hole kinematics and the laws of ordinary thermodynamics, but has been so far considered only in an asymptotically flat case. *Does such the correspondence rely strongly on the feature of the gravity vanishing at the infinity?* In order to resolve this question, it should be considered for the first to extend the Hawking radiation to a case with a dynamical boundary condition like an expanding universe. Therefore the Hawking radiation in an expanding universe is discussed in this paper. As a concrete model of a black hole in an expanding universe, we use the swiss cheese universe which is the spacetime including a Schwarzschild black hole in the Friedmann-Robertson-Walker universe. Further for simplicity, our calculation is performed in two dimension. The resultant spectrum of the Hawking radiation measured by a comoving observer is generally different from a thermal one. We find that the qualitative behavior of the non-thermal spectrum is of dumping oscillation as a function of the frequency measured by the observer, and that the intensity of the Hawking radiation is enhanced by the presence of a cosmological expansion. It is appropriate to say that *a black hole with an asymptotically flat boundary condition stays in a lowest energy thermal equilibrium state, and that, once a black hole is put into an expanding universe, it is excited to a non-equilibrium state and emits its mass energy with stronger intensity than a thermal one.*

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1 Introduction

So far various properties of black hole spacetimes have been revealed. Among so many well-known properties of the black hole, one of the most remarkable ones is the black hole thermodynamics. This tells us a nice correspondence between the classical properties of a black hole spacetime and the laws of ordinary thermodynamics [2]. The black hole thermodynamics is one of the most important stages for understanding the nature of a strong gravity. However this nice correspondence has been considered only in the asymptotically flat case. *Does such the correspondence rely strongly on the feature of the gravity vanishing at the infinity?* It has not been clarified if such the correspondence is true of the case with a dynamical boundary condition. [1] My general purpose is to *extend the black hole thermodynamics to a dynamical situation, and hopefully find a non-gravitational system which corresponds to a black hole with a dynamical boundary condition like an expanding universe.* By the way, the Hawking radiation plays the essential role in the black hole thermodynamics to determine precisely the temperature of a black hole with asymptotically flatness [3]. Hence, as the first step to attack my purpose, this paper is especially designed for the Hawking radiation in an expanding universe.

There is a serious problem of defining a black hole. A black hole is defined as the spacetime region excluded from the causal past of the future null infinity [4]. The future null infinity for the asymptotically flat case is well known, therefore it is rather clear if an asymptotically flat spacetime includes a black hole. On the other hand, however, under a dynamical boundary condition, it is generally quite difficult to extract the asymptotic structure of the spacetime directly from the Einstein equation. The black hole cannot be defined unless the asymptotic structure is known. In order to avoid the problem of defining the black hole in an expanding universe, we make use of the *swiss cheese universe.*

The swiss cheese universe (SCU) is the spacetime including a Schwarzschild black hole in a Friedmann-Robertson-Walker (FRW) universe. It is not the solution obtained by solving directly the Einstein equation, but the solution constructed by connecting the Schwarzschild spacetime with the FRW universe at a given spherically symmetric timelike hypersurface Σ by the Israel junction condition with requiring no surface energy density confined on Σ [5] [6]. The spatial section of Σ which is a two dimensional sphere is expanding as seen from the black hole side with respect to the

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[1]It is not too much to say that only a few studies have been done on the issue of a black hole in some dynamical background [1].

Schwarzschild coordinate. So once the two sphere on Σ was placed outside the gravitational radius $R_g = 2GM$ at a given time, the event horizon should last to exist after that time. That is, the existence of a black hole is guaranteed in the SCU.

Another technical problem has still been left. Even if with the asymptotic flatness, the curvature scattering makes it very difficult to obtain a complete form of the Hawking spectrum in four dimensional case. Indeed, the curvature scattering is ignored in the statement that an asymptotically flat black hole is in a thermal equilibrium state. So we also want to ignore the curvature scattering for calculating the Hawking radiation in the SCU. By the way, the curvature scattering for a minimal coupling massless scalar field vanishes on any two dimensional spacetimes due to the conformal flatness of the spacetime. Therefore in this paper, we treat the two dimensional SCU for simplicity, and introduce the minimal coupling massless scalar field for calculating the Hawking radiation.

In section 2, we briefly explain the two dimensional "eternal" swiss cheese universe and show the strategy to obtain the Hawking radiation on two dimensional SCU. The behavior of a minimal coupling massless scalar field on the eternal SCU is analyzed in section 3. Section 4 is devoted to computing the Hawking radiation. Summary and discussions are given in section 5.

The unit throughout this paper is $c = \hbar = k_B = 1$. The metric signature is $(-, +, +, +)$ for four dimension and similarly for two dimension.

2 "Eternal" swiss cheese universe and the strategy

2.1 Two dimensional "eternal" SCU

The two dimensional SCU should be constructed by connecting the Schwarzschild spacetime with the FRW universe at a given timelike surface (curve) Σ. The way of junction is very similar to the four dimensional case explained in appendix A. Hereafter as the terminology, let the word "BH-side" denote the spacetime region inside Σ where the metric is given by (1) or (2), and "FRW-side" for the region outside Σ where the metric is (6) or (7). If a quantity Q is measured in the BH-side, we express it as Q_{BH} and similarly Q_F in the FRW-side.

The metric of the BH-side is given by

$$
\begin{aligned}
ds_{BH}^2 &= -\left(1 - \frac{R_g}{R}\right) dT^2 + \left(1 - \frac{R_g}{R}\right)^{-1} dR^2 \quad \text{, in TR-system} \\
&= \frac{4R_g^3}{R} e^{-R/R_g} \left[-dW^2 + dX^2\right] \quad \text{, in WX-system},
\end{aligned}
$$

(1)

(2)

where $R_g = 2GM$, G is the gravitational constant, and the meaning of M is given at the end of this paragraph. The terms "TR-system" and "WX-system" mean respectively the Schwarzschild coordinate and the Kruskal-Szekeres coordinate in the BH-side. Because the maximally extended Schwarzschild spacetime has two exterior regions of a black hole, there can also be two exterior regions in the BH-side of the SCU. We distinguish them by calling "L-region" and "R-region". The coordinate transformation between the TR-system and the WX-system is given by

$$
W - X = \mp \exp[-\kappa(T - R^\star)] \quad , \quad W + X = \pm \exp[\kappa(T + R^\star)],
$$

(3)

where the upper signature is for the R-region and the lower for the L-region, $R^\star = R + R_g \ln(R/R_g - 1)$ and $\kappa = 1/2R_g$. We set the Killing vector $\tilde{\xi}_{BH}$ in the WX-system as

$$
\tilde{\xi}_{BH} = \partial_W \quad \text{, all over the BH-side},
$$

(4)

where a tilde is added to the quantity measured in the WX-system. Adopting the direction of this Killing vector as the standard of the future direction, the Killing vector ξ_{BH} in the TR-system is given by

$$
\xi_{BH} = \begin{cases} \partial_T & \text{, in the R-region} \\ -\partial_T & \text{, in the L-region}. \end{cases}
$$

(5)

The normalization of the vectors (4) and (5) is determined at the asymptotically flat region of the full Schwarzschild spacetime which is not present in the SCU. However once we accept them as the timelike Killing vector, κ can be interpreted as the surface gravity of the event horizon, and further M becomes exactly equal to the Komar mass [7] evaluated on any spatial section of Σ. Such a Komar mass [7] defined using our Killing vector (4) or (5) is invariant under the deformation of the spatial section of Σ. See the end of appendix A for more detailed explanation of κ and M.

The metric of the FRW-side in the comoving coordinate is

$$
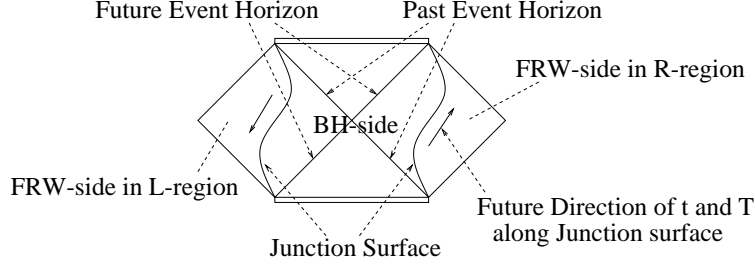ds_F^2 = -dt^2 + a(t)^2 \frac{dr^2}{1 - kr^2},
$$

(6)

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Figure 1: Two dimensional eternal swiss cheese universe. The junction surface is set outside the gravitational radius so that a Black hole exists eternally in the BH-side.

where $k = \pm 1, 0$ is the spatial curvature, and $a(t)$ is the scale factor. The comoving spatial coordinate in two dimensional case can range over infinity $-\infty < r < \infty$ for open and flat cases $k = -1, 0$. There is coordinate singularities at $r = \pm 1$ for closed case $k = 1$, but it can be eliminated by the transformation from the comoving coordinate (t, r) to the conformal coordinate $(\eta, r^\star)$ which is given by $d\eta = dt/a(t)$ and $r = \sin r^\star$, $r^\star$, or $\sinh r^\star$ for $k = 1, 0$ or -1 respectively. Then the metric becomes

$$ds_F^2 = a(\eta)^2 \left(-d\eta^2 + dr^{\star\,2} \right) , \tag{7}$$

where $-\infty < r^\star < \infty$. Hereafter as the terminology we call the conformal coordinate the Cfl-system. The comoving coordinate is equivalent to the Cfl-system in the sense that there exists one comoving coordinate point (t, r) for each point of the Cfl-system $(\eta, r^\star)$, where this correspondence is the one-to-one mapping for open and flat cases and the onto-mapping for closed case.

The two dimensional FRW metric does not have any timelike Killing vector, but has a conformal timelike Killing vector ξ_F which satisfies $\xi_{F\,\mu;\nu} + \xi_{F\,\mu;\nu} \propto g_{\mu\nu}$,

$$\xi_F = \begin{cases} \partial_\eta & \text{, in the R-region} \\ -\partial_\eta & \text{, in the L-region} . \end{cases} \tag{8}$$

In order to treat the two dimensional SCU as a simplification of four dimensional case, we require the same kinematics of the junction surface and the same relation between the temporal coordinates t and T as four dimensional case shown in the appendix A. That is, the location of Σ is given by

$$\Sigma : \begin{cases} R(t) = a(t)\, r_0 & \text{, in the BH-side,} \\ r = r_0 (= \text{const.}) & \text{, in the FRW-side,} \end{cases} \tag{9}$$

and the temporal coordinates are related by

$$\frac{dT(t)}{dt} = \frac{\sqrt{1 - k\, r_0^2}}{1 - R_g/R(t)} . \tag{10}$$

The unit normal vector to Σ is given by the same form as the four dimensional case (63) and (64)

$$n_F^\mu = \left(0 \, , \, \frac{1}{a(t)} \right) \quad \text{in Cfl-system,} \tag{11}$$

$$n_{BH}^\mu = \left(\frac{\dot{R}(t)}{1 - R_g/R(t)} \, , \, \left(1 - \frac{R_g}{R(t)} \right) \dot{T}(t) \right) \quad \text{in TR-system,} \tag{12}$$

where $\dot{Q} = dQ/dt$. Note that the equation (10) and the normalization of (12), $\mathbf{n}_{BH}^2 = 1$, gives the Friedmann equation of dust matter (66). However in order to extract the essence of the Hawking radiation in an expanding universe in a simple way, we dear to assume that the scale factor satisfies $a(t)\, r_0 > R_g$ for $-\infty < t < \infty$. That is, we consider the two dimensional SCU in which a black hole can exist eternally. The conformal diagram of such the "eternal" SCU is shown at the figure 1. With this assumption, a black hole can be defined without respect to the spatial curvature of the FRW-side, and the event horizon bifurcates. Hereafter throughout this paper, we adopt the eternal SCU as the background spacetime. The validity of the eternal SCU will be discussed at the end of section 5.

2.2 Strategy for the Hawking radiation

2.2.1 Choice of the Vacuum state

As discussed in section 1, we introduce a minimal coupling massless scalar field Φ and neglect any back reaction of Φ to the background eternal SCU. The purpose of this paper is to compute the Hawking radiation received by a comoving observer in the FRW-side. Then, because of the equivalence between the comoving coordinate and the Cfl-system, the expectation value of the number operator of the quantized scalar field in the Cfl-system attracts our interest.

Here we have to specify on which vacuum state the expectation value should be calculated. For the asymptotically flat black hole spacetime, there are two candidates for such the vacuum state: the Hartle-Hawking state [8] and the Unruh state [9]. The former describes that a black hole is surrounded by a thermal cavity of temperature $\kappa/2\pi$ and the whole system is in equilibrium. On the other hand, the latter corresponds to the situation that the environment surrounding the black hole is not a thermal cavity but an empty space, that is, the black hole is evaporating due to the Hawking radiation without absorbing any energy from the empty environment. By the way, it is not *a priori* known whether the black hole in the SCU emits a thermal radiation or not. Therefore it seems appropriate for the SCU not to put the black hole in a thermal cavity, but to put in an empty environment. Then we search what kind of spectrum is radiated out of the black hole in the SCU. From above considerations, we adopt the Unruh state as the vacuum state on which the Hawking radiation is computed. Further due to the presence of the junction surface, we use the following strategy of computing the Hawking radiation.

2.2.2 First step: construction of the mode functions

The vacuum state is constructed by quantizing the scalar field Φ. In order to quantize Φ, we have to obtain the positive frequency mode functions. On the SCU, there are two categories of mode functions: one of them is a collection of the mode solutions which is monochromatic in the BH-side but a superposition of some monochromatic modes in the FRW-side because of the junction conditions of Φ, (68) and (69) given in appendix B. Another category is of the monochromatic mode in the FRW-side but a superposition in the BH-side. And note that a mode function generally takes a different functional form according to the coordinate system in which the Klein-Gordon equation is solved. Then we search the following three modes;

1. Monochromatic mode in BH-side with respect to the WX-system, which we call the WX-mode

2. Monochromatic mode in BH-side with respect to the TR-system, which we call the TR-mode

3. Monochromatic mode in FRW-side with respect to the Cfl-system, which we call the Cfl-mode

Because our background is two dimensional and Φ is massless and has no coupling to gravity, the curvature scattering of Φ never occur. Therefore what we have to take care is the possibility of the reflection of Φ at Σ. Further as seen in the next section, it will be revealed that no reflection at Σ arises due to the special property of our junction conditions, and we can accomplish completely the construction of these modes. Then we define the positive frequency WX-mode, TR-mode and Cfl-mode, with respect to the (conformal) timelike Killing vectors (4), (5) and (8). That is, the positive frequency mode should satisfy $\mathcal{L}_\xi \Phi = -i\omega\Phi$, where $\omega(>0)$ is the frequency.

2.2.3 Second step: quantization of the scalar field

We quantize the minimal coupling scalar field Φ in the WX-system, the TR-system and the Cfl-system using the mode functions obtained in the first step. Then the vacuum states of each mode functions are defined.

The observer is comoving in the FRW-side, so the vacuum state appropriate for describing the empty environment surrounding the black hole is that of the Cfl-mode. Therefore the Unruh state for the SCU should be composed of two vacuum states: one of them is the vacuum of the WX-mode $|0\rangle_W$ on the past event horizon and another is that of the Cfl-mode $|0\rangle_\eta$ on the past null infinity. [2]

By the way consider if the background is four dimensional, then it is very difficult to solve the Klein-Gordon equation on the whole spacetime due to the curvature scattering. However near the event horizon and the null infinity, the solution can be found. Therefore for four dimensional case, we cannot help using the Unruh state as our vacuum state on which the Hawking radiation is calculated. However on the two dimensional eternal SCU, the exact form of the mode function can be constructed on the whole spacetime in the first step. So due to the absence of the curvature scattering and the reflection at Σ, the out-going modes radiated from the past event horizon are received by the comoving observer without loss of the energy of the out-going flux, and the in-going mode from the past null infinity never reach the observer. That is, the vacuum $|0\rangle_\eta$ on the past null infinity has no effect on the Hawking radiation received by the comoving observer. Consequently it is enough for us to prepare the vacuum state $|0\rangle_W$ at one Cauchy surface as the vacuum state on which the Hawking radiation is calculated. Hence in the following sections, we do not take care about the vacuum $|0\rangle_\eta$ in the

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[2]For asymptotically flat case, the observer rests on $R = $ constant, so the vacuum of the TR-mode is chosen on the past null infinity.
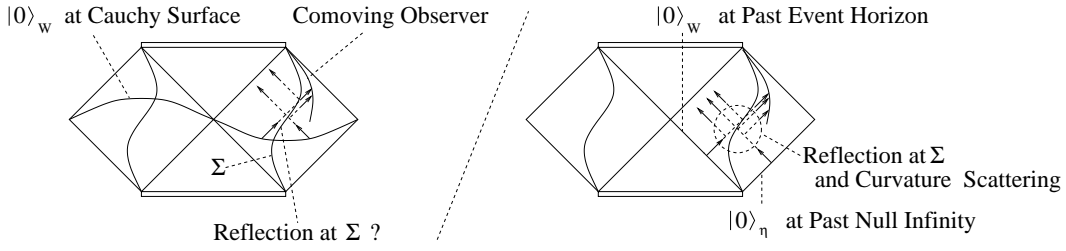
Figure 2: Right figure shows the vacuum state for four dimensional case. Right one is of two dimensional case. In section 3 it will be found that no reflection takes place for two dimensional case, and it is enough to prepare the vacuum $|0\rangle_W$ on a Cauchy surface for computing the Hawking radiation in the two dimensional SCU.

Unruh state, and aim to calculate just the Bogoljubov transformation from the WX-mode to the Cfl-mode. This can be understood as the particle of the Cfl-mode created in the vacuum state of the WX-mode.

To proceed this calculation, we make use of another intermediate transformation, that is, we perform the successive Bogoljubov transformation from the WX-mode to the TR-mode, and from the TR-mode to the Cfl-mode. Using this successive transformation, the difference between the asymptotically flat case and our SCU case should be clarified. The first transformation from the WX-mode to the TR-mode gives the thermal Hawking spectrum since the bifurcate event horizon of the eternal SCU lets us to make the same discussion given in the reference [9]. Then the deviation from the asymptotically flat case should arise from the second transformation from the TR-mode to the Cfl-mode.

3 Massless scalar field in the eternal SCU

3.1 Classical mode function

3.1.1 Preparations

We seek the positive frequency TR-mode, WX-mode and Cfl-mode on the SCU. However a positive frequency mode is out-going for positive wave number, and in-going for negative wave number. Therefore we look for the out-going and in-going modes. Those modes can be constructed by connecting the mode function obtained on the full Schwarzschild spacetime with that obtained on the full FRW spacetime at Σ. Because Φ is a minimal coupling massless field, it is natural to require that no potential of Φ is confined at Σ, that is, the field equation of the form $\Box\Phi = 0$ holds even at Σ. This requirement determines the junction conditions of Φ at Σ,

$$\Phi_{BH}|_\Sigma = \Phi_{FRW}|_\Sigma \quad , \quad \partial_n \Phi_{BH}|_\Sigma = \partial_n \Phi_{FRW}|_\Sigma , \tag{13}$$

where $\partial_n = n^\mu \partial_\mu$ and n^μ is the unit normal to Σ, (11) or (12). The detailed derivation of these conditions is given in the appendix B.

The mode functions satisfying $\Box\Phi = 0$ on the full Schwarzschild spacetime are,

$$\phi^{(\pm)}_{BH,\Omega} = \frac{1}{\sqrt{4\pi|\Omega|}} \times \begin{cases} e^{-i\Omega(T-R^\star)} & ,\text{out-going} \\ e^{-i\Omega(T+R^\star)} & ,\text{in-going} \end{cases} \quad \text{in TR-system}, \tag{14}$$

$$\tilde\phi^{(\pm)}_{BH,\tilde\Omega} = \frac{1}{\sqrt{4\pi|\tilde\Omega|}} \times \begin{cases} e^{-i\tilde\Omega(W-X)} & ,\text{out-going} \\ e^{-i\tilde\Omega(W+X)} & ,\text{in-going} \end{cases} \quad \text{in WX-system}, \tag{15}$$

where the upper suffix "+" denotes the out-going mode and the lower "−" is for the in-going one, and the normalization factor is determined by an ordinary inner product of the full Schwarzschild spacetime. The mode functions on the full FRW spacetime are

$$\phi^{(1,2)}_{F,\omega} = \frac{1}{\sqrt{4\pi|\omega|}} \times \begin{cases} e^{-i\omega(\eta-r^\star)} & ,\text{out-going} \\ e^{-i\omega(\eta+r^\star)} & ,\text{in-going} \end{cases} \quad \text{in Cfl-system}, \tag{16}$$

where the upper suffix "1" denotes the out-going mode and the lower "2" is for the in-going one, and the normalization factor is determined by an ordinary inner product. Even if we solve $\Box\Phi = 0$ using the comoving coordinate, the same mode function is obtained. For later convenience, we introduce the notations:

$$f_{F,\omega}(\eta) \equiv e^{-i\omega\eta} \quad \text{and} \quad h^{(1,2)}_{F,\omega}(r^\star) \equiv \frac{1}{\sqrt{4\pi|\omega|}} e^{\pm i\omega r^\star} . \tag{17}$$

5

Note that, because of the assumption of the eternal SCU, the temporal part $f_{F,\omega}(\eta)$ can span a complete orthogonal set with respect to $\eta \in (-\infty, +\infty)$ and $\omega \in (-\infty, +\infty)$ as

$$\int_{-\infty}^{\infty} d\eta\, f_{F,\omega}^*(\eta)\, f_{F,\omega'}(\eta) = 2\pi\, \delta(\omega - \omega') \quad , \quad \int_{-\infty}^{\infty} d\omega\, f_{F,\omega}^*(\eta)\, f_{F,\omega}(\eta') = 2\pi\, \delta(\eta - \eta')\,. \tag{18}$$

Referring to the ordinary definition of the inner products on the full Schwarzschild and FRW spacetimes, we define that on the SCU,

$$(\psi_1,\, \psi_2) = i \int_{-\infty}^{R^\star(\eta)} dR^\star\, \psi_1^* \overset{\leftrightarrow}{\partial_p} \psi_2 + i \int_{r_0^\star}^{\infty} dr^\star\, \psi_1^* \overset{\leftrightarrow}{\partial_p} \psi_2\,, \tag{19}$$

where ψ denotes the solution of $\Box\Phi = 0$ on the SCU, $\partial_p = (1/|\xi|)\xi^\mu\partial_\mu$, ξ denotes the (conformal) Killing vectors (8) and (5) or (4), and $r_0^\star$ is the coordinate value of $r^\star$ at the junction surface Σ. This inner product is defined on the spatial surface of $\eta = $ constant in the SCU.

3.1.2 Out-going and in-going TR-modes and WX-modes

TR-mode

As mentioned at the subsection 2.2.2, the out-going and in-going TR-modes are monochromatic in the BH-side but a superposition of some monochromatic modes in the FRW-side. Then we can express the out-going TR-mode $\psi_{BH,\Omega}^{(+)}$ and the in-going one $\psi_{BH,\Omega}^{(-)}$ as

$$\psi_{BH,\Omega}^{(+)} = \begin{cases} \phi_{BH,\Omega}^{(+)} & \text{, in BH-side} \\ \int_{-\infty}^{\infty} d\omega \left[A_{\Omega\omega}^{(1)}\, \phi_{F,\omega}^{(1)} + A_{\Omega\omega}^{(2)}\, \phi_{F,\omega}^{(2)} \right] & \text{, in FRW-side}\,, \end{cases} \tag{20}$$

$$\psi_{BH,\Omega}^{(-)} = \begin{cases} \phi_{BH,\Omega}^{(-)} & \text{, in BH-side} \\ \int_{-\infty}^{\infty} d\omega \left[B_{\Omega\omega}^{(1)}\, \phi_{F,\omega}^{(1)} + B_{\Omega\omega}^{(2)}\, \phi_{F,\omega}^{(2)} \right] & \text{, in FRW-side}\,, \end{cases} \tag{21}$$

where the junction coefficients $A^{(1;2)}$ and $B^{(1;2)}$ are determined by the junction conditions (13), which become

$$\phi_{BH,\Omega}^{(+)}(T(\eta), R(\eta)) = \int_{-\infty}^{\infty} d\omega \left[A_{\Omega\omega}^{(1)}\, f_{F,\omega}(\eta)\, h_{F,\omega}^{(1)}(r_0) + A_{\Omega\omega}^{(2)}\, f_{F,\omega}(\eta)\, h_{F,\omega}^{(2)}(r_0) \right]\,,$$

$$\partial_n \phi_{BH,\Omega}^{(+)}(T(\eta), R(\eta)) = \frac{1}{a(\eta)} \int_{-\infty}^{\infty} d\omega \left[A_{\Omega\omega}^{(1)}\, f_{F,\omega}(\eta)\, \frac{d\,h_{F,\omega}^{(1)}}{dr^\star}(r_0) + A_{\Omega\omega}^{(2)}\, f_{F,\omega}(\eta)\, \frac{d\,h_{F,\omega}^{(2)}}{dr^\star}(r_0) \right]\,, \tag{22}$$

and the similar equations for $B^{(1;2)}$ holds, where $(T(\eta), R(\eta))$ is the coordinate point on Σ in the TR-system, and note that $(dh_F/dr^\star)|_{r=r_0} = $ constant.

Here in this subsection, let us calculate to obtain $A^{(1;2)}$, then the similar calculation gives $B^{(1;2)}$. Using the completeness (18), we obtain the integral representations

$$A_{\Omega\omega}^{(1)} = \operatorname{sgn}(\omega)\, \frac{e^{-i\omega r_0^\star}}{4\pi\sqrt{|\Omega\omega|}}\, [I_A(\Omega,\omega) - J_A(\Omega,\omega)]$$

$$A_{\Omega\omega}^{(2)} = \operatorname{sgn}(\omega)\, \frac{e^{i\omega r_0^\star}}{4\pi\sqrt{|\Omega\omega|}}\, [I_A(\Omega,\omega) + J_A(\Omega,\omega)]\,, \tag{23}$$

where $\operatorname{sgn}(\omega) = \omega/|\omega|$. Here I_A and J_A are given by

$$I_A(\Omega,\omega) = \omega \int_{-\infty}^{\infty} d\eta\, e^{i\omega\eta - i\,\Omega[T(\eta) - R^\star(\eta)]} \quad , \quad J_A(\Omega,\omega) = \Omega \int_{-\infty}^{\infty} d\eta\, P_A(\eta)\, e^{i\omega\eta - i\,\Omega[T(\eta) - R^\star(\eta)]}\,, \tag{24}$$

where

$$P_A(\eta) = \left(\frac{a'(\eta)\, r_0}{a(\eta)\sqrt{1 - k\, r_0^2}} - 1 \right) \frac{dT(\eta)}{d\eta}\,,$$

and $a' = da/d\eta$. Once the scale factor is specified, the junction coefficients can be determined in principle. One may think that the out-going TR-mode is a superposition of the out-going and in-going modes in the FRW-side, and so is the in-going TR-mode. However it is not the case. To show it, it is essential to notice that

$$\frac{d}{d\eta}\, e^{i\omega\eta - i\,\Omega[T(\eta) - R^\star(\eta)]} = i\, [\omega + \Omega P_A(\eta)]\, e^{i\omega\eta - i\,\Omega[T(\eta) - R^\star(\eta)]}\,, \tag{25}$$

which gives

$$I_A + J_A = -i \ e^{i\omega\eta - i\,\Omega[T(\eta) - R^\star(\eta)]}\Big|_{\eta=-\infty}^{\eta=\infty}.$$ (26)

Then we obtain the out-going TR-mode in the FRW-side to be

$$\psi_{BH,\Omega}^{(+)} \text{ in FRW-side} = \int_{-\infty}^{\infty} d\omega \ \left(A_{\Omega\omega}^{(1)}\,\phi_{F,\omega}^{(1)}\right) + \text{sgn}(\bar{\eta})\,\frac{\pi}{(4\pi)^{3/2}\sqrt{|\Omega|}}\ e^{-i\,\Omega[T(\bar{\eta}) - R^\star(\bar{\eta})]}\Big|_{\bar{\eta}=-\infty}^{\bar{\eta}=\infty},$$ (27)

where $\int_{-\infty}^{\infty} dx\,(1/x)\,\exp(\pm ix) = \pm i\pi$ is used. As discussed in the appendix C, the second term should be zero, and we can find $I_A = -J_A$ due to the equations (26).

The same way of calculation done so far holds for the other junction coefficients $B^{(1,2)}$. Then the out-going and in-going TR-modes are obtained to be

$$\psi_{BH,\Omega}^{(+)} = \begin{cases} \phi_{BH,\Omega}^{(+)} & \text{, in BH-side} \\ \int_{-\infty}^{\infty} d\omega\, A_{\Omega\omega}^{(1)}\,\phi_{F,\omega}^{(1)} & \text{, in FRW-side}, \end{cases}$$ (28)

$$\psi_{BH,\Omega}^{(-)} = \begin{cases} \phi_{BH,\Omega}^{(-)} & \text{, in BH-side} \\ \int_{-\infty}^{\infty} d\omega\, B_{\Omega\omega}^{(2)}\,\phi_{F,\omega}^{(2)} & \text{, in FRW-side}, \end{cases}$$ (29)

where $A^{(1)}$ and $B^{(2)}$ are summarized as

$$A_{\Omega\omega}^{(1)} = \text{sgn}(\omega)\,\frac{e^{-i\omega r_0^\star}}{2\pi\sqrt{|\Omega\omega|}}\,I_A(\Omega,\omega) = \frac{e^{-i\omega r_0^\star}}{2\pi}\sqrt{\left|\frac{\omega}{\Omega}\right|}\int_{-\infty}^{\infty} d\eta\, e^{i\omega\eta - i\,\Omega[T(\eta)-R^\star(\eta)]},$$ (30)

and

$$B_{\Omega\omega}^{(2)} = \frac{e^{i\omega r_0^\star}}{2\pi}\sqrt{\left|\frac{\omega}{\Omega}\right|}\int_{-\infty}^{\infty} d\eta\, e^{i\omega\eta - i\,\Omega[T(\eta)+R^\star(\eta)]}.$$

These results mean that the junction surface does not reflect the scalar field, therefore the out-going TR-mode is a superposition of the out-going modes in the FRW-side without including any in-going mode, and the in-going TR-mode consists of the in-going modes in the FRW-side. Further we can check by some routine computations that the inner products of these modes defined by (19) result in

$$\left(\psi_{BH,\Omega}^{(\pm)},\,\psi_{BH,\Omega'}^{(\pm)}\right) = \text{sgn}(\Omega)\,\delta(\Omega - \Omega') \quad, \quad \left(\psi_{BH,\Omega}^{(\pm)},\,\psi_{BH,\Omega'}^{(\mp)}\right) = 0.$$ (31)

This means that the out-going and in-going TR-modes given by (28) and (29) span a complete orthonormal basis.

There are two reasons for the disappearance of reflection at Σ: (i) the junction condition for the scalar field with no surface potential at Σ, and (ii) the conformal flatness of the background spacetime. To understand the first reason, let us consider the case of $F \neq 0$ for the junction condition (69) in the appendix B. In this case we can follow the same way of calculation to obtain a similar result to (23),

$$A_{\Omega\omega}^{(1)} = \text{(the same form as (23))} + \text{sgn}(\omega)\,2\pi\,i\,h_{F,\omega}^{(2)}\int_{-\infty}^{\infty} d\eta\, a\, f_{F,\omega}^*\, F$$

$$A_{\Omega\omega}^{(2)} = \text{(the same form as (23))} - \text{sgn}(\omega)\,2\pi\,i\,h_{F,\omega}^{(1)}\int_{-\infty}^{\infty} d\eta\, a\, f_{F,\omega}^*\, F.$$

The second term of $A^{(2)}$ does not vanish, and the reflection should take place at Σ. On the other hand even if $F = 0$, when the background spacetime is not conformal flat, $A^{(2)}$ cannot vanish due to the curvature scattering.

WX-mode

The out-going and in-going WX-modes $\tilde{\psi}_{BH,\tilde{\Omega}}^{(\pm)}$ can be obtained by a similar way to the TR-mode. The resultant WX-mode is not reflected at Σ too,

$$\tilde{\psi}_{BH,\tilde{\Omega}}^{(+)} = \begin{cases} \tilde{\phi}_{BH,\tilde{\Omega}}^{(+)} & \text{, in BH-side} \\ \int_{-\infty}^{\infty} d\omega\, \tilde{A}_{\tilde{\Omega}\omega}^{(1)}\,\phi_{F,\omega}^{(1)} & \text{, in FRW-side}, \end{cases}$$ (32)

$$\tilde{\psi}_{BH,\tilde{\Omega}}^{(-)} = \begin{cases} \tilde{\phi}_{BH,\tilde{\Omega}}^{(-)} & \text{, in BH-side} \\ \int_{-\infty}^{\infty} d\omega\, \tilde{B}_{\tilde{\Omega}\omega}^{(2)}\,\phi_{F,\omega}^{(2)} & \text{, in FRW-side}. \end{cases}$$ (33)

The junction coefficients $\tilde{A}^{(1)}$ and $\tilde{B}^{(2)}$ are given by

$$\tilde{A}^{(1)}_{\tilde{\Omega}\omega} = \frac{e^{-i\omega r_0^\star}}{2\pi} \sqrt{\left|\frac{\omega}{\tilde{\Omega}}\right|} \int_{-\infty}^{\infty} d\eta \, e^{i\omega\eta - i\tilde{\Omega}[W(\eta) - X(\eta)]}$$

$$\tilde{B}^{(2)}_{\tilde{\Omega}\omega} = \frac{e^{i\omega r_0^\star}}{2\pi} \sqrt{\left|\frac{\omega}{\tilde{\Omega}}\right|} \int_{-\infty}^{\infty} d\eta \, e^{i\omega\eta - i\tilde{\Omega}[W(\eta) + X(\eta)]},$$

where $(W(\eta), X(\eta))$ is the coordinate of a point on Σ measured in the WX-system. These WX-modes $\tilde{\psi}^{(\pm)}_{BH,\tilde{\Omega}}$ also span a complete orthonormal basis.

3.1.3   Out-going and in-going Cfl-modes

The out-going and in-going Cfl-modes $\psi^{(1,2)}_{F,\omega}$ are monochromatic in the FRW-side and a superposition of some monochromatic modes in the BH-side. Because the reflection of Φ at Σ disappears as discussed in the previous subsection, these modes can be expressed as

$$\psi^{(1)}_{F,\omega} = \begin{cases} \int_{-\infty}^{\infty} d\Omega \, C^{(+)}_{\Omega\omega} \phi^{(+)}_{BH,\Omega} & \text{, in BH-side} \\ \phi^{(1)}_{F,\omega} & \text{, in FRW-side}, \end{cases} \tag{34}$$

$$\psi^{(2)}_{F,\omega} = \begin{cases} \int_{-\infty}^{\infty} d\Omega \, D^{(-)}_{\Omega\omega} \phi^{(-)}_{BH,\Omega} & \text{, in BH-side} \\ \phi^{(2)}_{F,\omega} & \text{, in FRW-side}. \end{cases} \tag{35}$$

where we use the TR-mode in the BH-side for later use. The junction coefficients $C^{(+)}$ and $D^{(-)}$ can be determined in principle by the junction conditions of the scalar field. However because the junction surface does not rest in the TR-system $R(\eta) = a(\eta) r_0$, it is technically difficult to follow the same calculation done in the previous subsection. In other words, the completeness of the temporal part, $e^{-i\Omega T}$, cannot extract the simple integral representation of the junction coefficients $C^{(+)}$ and $D^{(-)}$ like (23). However the analytic representations of them will be obtained in next section 3.2 with making use of the Bogoljubov transformation. Therefore we assume temporarily that these Cfl-modes span a complete orthonormal basis. After obtaining the integral representations of $C^{(+)}$ and $D^{(-)}$, we will find that this assumption is consistent.

We can easily find some relations between $A^{(1)}$, $B^{(2)}$ and $C^{(+)}$, $D^{(-)}$. By substituting (28) into (34) and other similar operations, we obtain

$$\int_{-\infty}^{\infty} d\omega \, A^{(1)}_{\Omega,\omega} C^{(+)}_{\Omega'\omega} = \delta(\Omega - \Omega') \qquad \int_{-\infty}^{\infty} d\Omega \, A^{(1)}_{\Omega,\omega} C^{(+)}_{\Omega\omega'} = \delta(\omega - \omega') \tag{36}$$

$$\int_{-\infty}^{\infty} d\omega \, B^{(2)}_{\Omega,\omega} D^{(-)}_{\Omega'\omega} = \delta(\Omega - \Omega') \qquad \int_{-\infty}^{\infty} d\Omega \, B^{(2)}_{\Omega,\omega} D^{(-)}_{\Omega\omega'} = \delta(\omega - \omega'). \tag{37}$$

3.2   Quantization

3.2.1   Vacuum states of WX-mode, TR-mode and Cfl-mode

Positive frequency modes

For canonical quantization of the scalar field Φ, we need the positive frequency mode. Hereafter as the notation, we denotes the positive frequency TR-mode, WX-mode and Cfl-mode respectively as Ψ_{BH}, $\tilde{\Psi}_{BH}$ and Ψ_F. These modes can be obtained from the out-going and in-going modes as

$$\Psi_{BH,K} = \begin{cases} \psi^{(+)}_{BH,\Omega=K} & \text{, for } K > 0 \\ \psi^{(-)}_{BH,\Omega=-K} & \text{, for } K < 0, \end{cases} \quad \text{(Positive Frequency TR-mode)}$$

where Ω is the frequency, K is the wave number and $\Omega = |K|$. Similarly we find

$$\tilde{\Psi}_{BH,K} = \begin{cases} \psi^{(+)}_{BH,\tilde{\Omega}=\tilde{K}} & \text{, for } \tilde{K} > 0 \\ \psi^{(-)}_{BH,\tilde{\Omega}=-\tilde{K}} & \text{, for } \tilde{K} < 0, \end{cases} \quad \text{(Positive Frequency WX-mode)}$$

and

$$\Psi_{F,k} = \begin{cases} \psi^{(1)}_{F,\omega=k} & \text{, for } k > 0 \\ \psi^{(2)}_{F,\omega=-k} & \text{, for } k < 0, \end{cases} \quad \text{(Positive Frequency Cfl-mode)}$$

where $\tilde{\Omega} = |\tilde{K}|$ and $\omega = |k|$. For example, the positive frequency TR-mode in the BH-side is expressed as

$$\Psi_{BH,K} \text{ in BH-side} = \frac{1}{\sqrt{4\pi|K|}} e^{-i(\Omega T - KR^\star)} = \frac{1}{\sqrt{4\pi|K|}} \times \begin{cases} e^{-iK(T-R^\star)} & \text{, for } K > 0 \\ e^{+iK(T+R^\star)} & \text{, for } K < 0 \,, \end{cases}$$

and this satisfies the definition of the positive frequency mode $\mathcal{L}_{\xi_{BH}}\Psi_{BH} = -i\Omega\Psi_{BH}$ where $\Omega > 0$.

We should note that the mode functions Ψ_{BH} and Ψ_F are defined only in the R-region or in the L-region of the maximally extended eternal SCU. Therefore we define the other mode functions as [9]

$$_R\Psi_{BH,K} = \begin{cases} \Psi_{BH,K} & \text{, in R-region} \\ 0 & \text{, in L-region} \end{cases} \quad , \quad _L\Psi_{BH,K} = \begin{cases} 0 & \text{, in R-region} \\ \Psi^*_{BH,K} & \text{, in L-region} \,, \end{cases}$$

and

$$_R\Psi_{F,k} = \begin{cases} \Psi_{F,k} & \text{, in R-region} \\ 0 & \text{, in L-region} \end{cases} \quad , \quad _L\Psi_{F,k} = \begin{cases} 0 & \text{, in R-region} \\ \Psi^*_{F,k} & \text{, in L-region} \,. \end{cases}$$

All of these modes are of the positive frequency, and can be defined on a Cauchy surface connecting the spatial infinities of R-region and L-region via the bifurcation point of the event horizon. We take the complex conjugate in the definition of $_L\Psi_{BH,K}$ and $_L\Psi_{F,k}$, since the future temporal direction in the L-region with respect to T and η is inverted in comparison with that in the R-region. Further it is useful to define the following mode functions [9],

$$\begin{aligned}
\hat{\Psi}^I_{BH,K} &= \frac{1}{|2\sinh(\pi\Omega/\kappa)|^{1/2}} \left[e^{\pi\Omega/2\kappa} \,_R\Psi_{BH,K} + e^{-\pi\Omega/2\kappa} \,_L\Psi^*_{BH,K} \right] \\
\hat{\Psi}^{II}_{BH,K} &= \frac{1}{|2\sinh(\pi\Omega/\kappa)|^{1/2}} \left[e^{-\pi\Omega/2\kappa} \,_R\Psi^*_{BH,K} + e^{\pi\Omega/2\kappa} \,_L\Psi_{BH,K} \right] \,.
\end{aligned} \tag{38}$$

These mode functions share the same analyticity as the mode $\tilde{\Psi}_{BH}$. That is, $\hat{\Psi}^{I,II}_{BH}$ and $\tilde{\Psi}_{BH}$ are analytic and bounded on the lower-half-plane of the complex $U(= W \pm X)$ plane.

Due to the completeness of the out-going and in-going TR-modes (31), the positive frequency TR-modes satisfy the complete orthonormal relations. So is the positive frequency WX-modes. For the positive frequency Cfl-modes, the assumption mentioned at the subsection 3.1.3 makes the positive frequency modes satisfy the complete orthonormal relations. Hence we can carry out the canonical quantization of Φ using each mode function.

Quantization with WX-mode

The scalar field Φ is quantized using the WX-mode as

$$\Phi = \int_{-\infty}^{\infty} d\tilde{K} \left[\tilde{a}_{\tilde{K}} \, \tilde{\Psi}_{BH,\tilde{K}} + \tilde{a}^\dagger_{\tilde{K}} \, \tilde{\Psi}^*_{BH,\tilde{K}} \right] \,.$$

Further because $\hat{\Psi}_{BH}$ and $\tilde{\Psi}_{BH}$ share the same analyticity, we can find another representation,

$$\Phi = \int_{-\infty}^{\infty} dK \left[a^I_K \, \hat{\Psi}^I_{BH,K} + a^{I\,\dagger}_K \, \hat{\Psi}^{I*}_{BH,K} + a^{II}_K \, \hat{\Psi}^{II}_{BH,K} + a^{II\,\dagger}_K \, \hat{\Psi}^{II*}_{BH,K} \right] \,.$$

Here $\tilde{a}_{\tilde{K}}$ and $a^{I,II}_K$ are the annihilation operators, and their Hermitian conjugates are the creation operators. Due to the same analyticity of $\hat{\Psi}^{I,II}_{BH}$ as $\tilde{\Psi}_{BH}$, they also share the same vacuum state $|0\rangle_W$ defined by

$$\tilde{a}_{\tilde{K}} \, |0\rangle_W = a^I_K \, |0\rangle_W = a^{II}_K \, |0\rangle_W = 0 \quad \text{for all } \tilde{K} \text{ and } K \,.$$

Quantization with TR-mode

The quantization of Φ using the TR-mode is represented as

$$\Phi = \int_{-\infty}^{\infty} dK \left[{}_R b_K \,_R\Psi_{BH,K} + {}_R b^\dagger_K \,_R\Psi^*_{BH,K} + {}_L b_K \,_L\Psi_{BH,K} + {}_L b^\dagger_K \,_L\Psi^*_{BH,K} \right] \,,$$

where $_R b_K$ and $_L b_K$ are the annihilation operators and $_R b^\dagger_K$ and $_L b^\dagger_K$ are the creation ones. The vacuum state of the TR-mode $|0\rangle_T$ is defined by

$$_R b_K \, |0\rangle_T = {}_L b_K \, |0\rangle_T = 0 \quad \text{for all } K \,.$$

Quantization with Cfl-mode

The quantization with the Cfl-mode is similar to that of the TR-mode,

$$\Phi = \int_{-\infty}^{\infty} dK \left[{}_R c_k \, {}_R \Psi_{F,k} + {}_R c_k^\dagger \, {}_R \Psi_{F,k}^* + {}_L c_k \, {}_L \Psi_{BH,K} + {}_L c_k^\dagger \, {}_L \Psi_{F,k}^* \right] ,$$

where ${}_R c_k$ and ${}_L c_k$ are the annihilation operators and ${}_R c_k^\dagger$ and ${}_L c_k^\dagger$ are the creation ones. The vacuum state of the Cfl-mode $|0\rangle_\eta$ is defined by

$$_R c_k \, |0\rangle_\eta = {}_L c_k \, |0\rangle_\eta = 0 \quad \text{for all } k \, .$$

3.2.2 Bogoljubov transformation

The Bogoljubov transformation is the change of positive frequency basis. Therefore the equation (38) gives the Bogoljubov transformation between the WX-mode and the TR-mode. In this subsection, we derive the Bogoljubov transformation between the TR-mode and the Cfl-mode. It is important to notice the implication of the relations (36) and (37) that, for the out-going modes,

$$
\begin{aligned}
\int_{-\infty}^{\infty} d\Omega \, C_{\Omega\omega}^{(+)} \, \psi_{BH,\Omega}^{(+)} &= \int_{-\infty}^{\infty} d\Omega \, C_{\Omega\omega}^{(+)} \, \phi_{BH,\Omega}^{(+)} = \psi_{F,\omega}^{(1)} \quad \text{in BH-side} \\
&= \iint_{-\infty}^{\infty} d\omega' \, d\Omega \, C_{\Omega\omega}^{(+)} \, A_{\Omega\omega'} \, \phi_{F,\omega'}^{(1)} = \psi_{F,\omega}^{(1)} \quad \text{in FRW-side} \, ,
\end{aligned}
$$

and a similar one for the in-going mode. This denotes that the junction coefficients $A^{(1)}$, $B^{(2)}$, $C^{(+)}$ and $D^{(-)}$ give the change of out-going and in-going basis as

$$\psi_{BH,\Omega}^{(+)} = \int_{-\infty}^{\infty} d\omega \, A_{\Omega\omega}^{(1)} \, \psi_{F,\omega}^{(1)} \quad , \quad \psi_{BH,\Omega}^{(-)} = \int_{-\infty}^{\infty} d\omega \, B_{\Omega\omega}^{(2)} \, \psi_{F,\omega}^{(2)} \, ,$$

and

$$\psi_{F,\omega}^{(1)} = \int_{-\infty}^{\infty} d\omega \, C_{\Omega\omega}^{(+)} \, \psi_{BH,\Omega}^{(+)} \quad , \quad \psi_{F,\omega}^{(2)} = \int_{-\infty}^{\infty} d\omega \, D_{\Omega\omega}^{(-)} \, \psi_{BH,\Omega}^{(-)} \, .$$

Consequently, by definition of Ψ_{BH}, Ψ_F, $\psi_{BH}^{(\pm)}$ and $\psi_F^{(1,2)}$, we obtain the Bogoljubov transformation between the positive frequency modes Ψ_{BH} and Ψ_F as

$$
\begin{aligned}
\Psi_{BH,K} &= \int_{-\infty}^{\infty} dk \left[\alpha_{Kk} \, \Psi_{F,k} + \beta_{Kk} \, \Psi_{F,k}^* \right] \\
\Psi_{F,k} &= \int_{-\infty}^{\infty} dK \left[\lambda_{Kk} \, \Psi_{BH,K} + \mu_{Kk} \, \Psi_{BH,K}^* \right] \, ,
\end{aligned}
$$

where the Bogoljubov coefficients are given as

$$
\begin{aligned}
\alpha_{Kk} &= \begin{cases} A_{Kk}^{(1)} & , \text{for } K > 0, \, k > 0 \\ 0 & , \text{for } K > 0 \text{ and } k < 0, \text{ or } K < 0 \text{ and } k > 0 \\ B_{-K-k}^{(2)} & , \text{for } K < 0, \, k < 0 \, , \end{cases} \\
\beta_{Kk} &= \begin{cases} A_{K-k}^{(1)} & , \text{for } K > 0, \, k > 0 \\ 0 & , \text{for } K > 0 \text{ and } k < 0, \text{ or } K < 0 \text{ and } k > 0 \\ B_{-Kk}^{(2)} & , \text{for } K < 0, \, k < 0 \, , \end{cases}
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
\lambda_{Kk} &= \begin{cases} C_{Kk}^{(+)} & , \text{for } K > 0, \, k > 0 \\ 0 & , \text{for } K > 0 \text{ and } k < 0, \text{ or } K < 0 \text{ and } k > 0 \\ D_{-K-k}^{(-)} & , \text{for } K < 0, \, k < 0 \, , \end{cases} \\
\mu_{Kk} &= \begin{cases} C_{-Kk}^{(+)} & , \text{for } K > 0, \, k > 0 \\ 0 & , \text{for } K > 0 \text{ and } k < 0, \text{ or } K < 0 \text{ and } k > 0 \\ D_{K-k}^{(-)} & , \text{for } K < 0, \, k < 0 \, . \end{cases}
\end{aligned}
$$

Further, taking the inverse of the Bogoljubov transformation, we can find the relation between (α, β) and (λ, μ) to be $\lambda_{Kk} = \alpha_{Kk}^*$, $\mu_{Kk} = -\beta_{Kk}^*$. Hence the Bogoljubov transformation from $({}_R\Psi_{BH,K}, {}_L\Psi_{BH,K})$ to $({}_R\Psi_{F,k}, {}_L\Psi_{F,k})$ is obtained to be

$$
\begin{aligned}
{}_R\Psi_{BH,K} &= \int_{-\infty}^{\infty} dk \left[\alpha_{Kk} \, {}_R\Psi_{F,k} + \beta_{Kk} \, {}_R\Psi_{F,k}^* \right] \\
{}_L\Psi_{BH,K} &= \int_{-\infty}^{\infty} dk \left[\alpha_{Kk}^* \, {}_L\Psi_{F,k} + \beta_{Kk}^* \, {}_L\Psi_{F,k}^* \right],
\end{aligned}
\tag{40}
$$

Finally note that the junction coefficients $C^{(+)}$ and $D^{(-)}$ are represented analytically using the other junction coefficients $A^{(1)}$ and $B^{(2)}$ through the Bogoljubov transformations obtained above. Therefore we can find that the out-going and in-going Cfl-modes span a complete orthonormal basis.

4 Hawking radiation in the SCU

4.1 Hawking radiation

We compute the Hawking radiation observed by a comoving observer. As discussed at the section 2.2, its spectrum is given by ${}_W\langle 0| N_{F,\omega} |0\rangle_W$, where $N_{F,\omega} = {}_Rc_\omega^\dagger \, {}_Rc_\omega$ is the number operator of particles of the Cfl-mode, [3] and $\omega (> 0)$ is the energy of the particle which is equivalent to the frequency of the positive frequency Cfl-mode. Further in order to treat the situation that the observer receives the out-going radiation from the black hole, we should consider only the out-going Cfl-mode, $k > 0$. Then taking the Bogoljubov transformations (38) and (40) successively, the Hawking spectrum is obtained to be

$$
{}_W\langle 0| N_{F,\omega} |0\rangle_W = \int_0^\infty d\Omega \, \frac{1}{e^{2\pi\Omega/\kappa} - 1} \, D_H(\Omega, \omega),
\tag{41}
$$

where D_H is given by

$$
D_H(\Omega, \omega) = \left| A_{\Omega\,\omega}^{(1)} \right|^2 + e^{2\pi\Omega/\kappa} \left| A_{\Omega\,-\omega}^{(1)} \right|^2 + 2\, e^{\pi\Omega/\kappa} \, \Re \left(A_{\Omega\,\omega}^{(1)} \, A_{\Omega\,-\omega}^{(1)} \right),
\tag{42}
$$

and $\Re$ denotes the real part, $A^{(1)}$ is given at (30), the relations $\Omega = |K| = K$ and $\omega = |k| = k$ are used, and due to the relations (39) the variable of integration is restricted to $K > 0 \Rightarrow \Omega > 0$.

Obviously the Hawking radiation in the SCU is totally different from a thermal spectrum. [4] The factor D_H represents such a difference, and we call D_H the *deviation factor*. Note that the particle creation due to the cosmological expansion does not take place with a minimal coupling massless scalar field in two dimensional background spacetime [10]. So the deviation factor D_H includes only the effect of the motion of the observer relative to the TR-system, and does not include the cosmological particle creation.

The other notice of the spectrum (41) is that the Bogoljubov transformations used in deriving the Hawking radiation are defined all over the spacetime. Especially whole of the temporal information from $\eta = -\infty$ to $\eta = \infty$ are included in this result. That is, we should interpret the observer is at a remote future $\eta \to +\infty$.

4.2 Divergence and normalization of the Hawking spectrum

The black hole should lose its mass energy M due to the Hawking radiation. However, since the back reaction to the background spacetime is ignored in our derivation of the resultant spectrum (41), the mass M remains constant during emitting the Hawking radiation. Thus, the spectrum (41) may give us the conclusion that the observer at a remote future receives an infinite energy from the black hole via the Hawking radiation. In the next subsection 4.2.1, we see the divergence of the spectrum (41) for the simplest case that the scale factor is constant. Then, the normalization of the spectrum (41) is discussed in the following subsection 4.2.2.

4.2.1 Simplest case: constant scale factor

Consider the case $a = a_c = $ constant, which should give the thermal spectrum. From (9), $R(\eta) = a_c r_0 = $ constant, then (10) gives $T(\eta) = b_c \, \eta + $ constant, where $b_c = a_c \sqrt{1 - k\, r_0{}^2}/(1 - R_g/a_c r_0)$. That is, $T(\eta) - R^\star(\eta) = b_c \, \eta - R_c^\star$,

---

[3] Here we set the observer in the R-region. But even if the observer is in the L-region, the resultant spectrum is the same.

[4] But the case that $a = $ constant should reproduce the thermal spectrum.

where $R_c^\star =$ constant. Then the junction coefficient $A_{\Omega\omega}^{(1)}$ given at (30) is calculated using the representation of the delta function $\int_{-\infty}^{\infty} dk \exp(ikx) = 2\pi\delta(x)$,

$$A_{\Omega\omega}^{(1)} = \sqrt{\left|\frac{\omega}{\Omega}\right|}\, e^{-i(\omega r_0^\star - \Omega R_c^\star)}\, \delta(\omega - b_c\Omega)\,. \tag{43}$$

Since $A_{\Omega-\omega}^{(1)} = 0$ for $\Omega > 0$ and $\omega > 0$, the deviation factor becomes $D_H = |A_{\Omega\omega}^{(1)}|^2$ which gives the spectrum (41) as,

$$_W\langle 0|\, N_{F,\omega}\, |0\rangle_W = \frac{b_c}{e^{2\pi\omega/b_c\kappa} - 1}\, \delta(0)\,. \tag{44}$$

This spectrum diverges by the delta function $\delta(0)$.

4.2.2   Finite time interval normalization

If the delta function appeared in (43) is removed, the divergence in the previous resultant spectrum (44) is normalized. Note that the delta function in (43) comes from the integral of infinite time interval in the representation (30). Therefore we make the comoving observer receive the Hawking radiation during a finite time interval, $\eta_i < \eta < \eta_f(= \eta_i + \eta_p)$. Further we introduce the periodic boundary condition for the scalar field, $\Phi(\eta + \eta_p) = \Phi(\eta)$, which changes the frequency of the Cfl-mode ω and that of the TR-mode Ω with the discrete quantities,

$$\omega = \frac{2\pi}{\eta_p}\, n \quad,\quad n = \pm 1, \pm 2, \pm 3, \cdots \quad \text{and} \quad \eta_p = \eta_f - \eta_i$$

$$\Omega = \frac{2\pi}{T_p}\, N \quad,\quad N = \pm 1, \pm 2, \pm 3, \cdots \quad \text{and} \quad T_p = T(\eta_f) - T(\eta_i)\,, \tag{45}$$

where, because the mode solution of zero frequency is trivial $\Phi = $ constant, the zero frequencies $n = 0$ and $N = 0$ are neglected. The integral with respect to ω, Ω and η should be replaced as

$$\int_{-\infty}^{\infty} d\omega \longrightarrow \sum_{n=-\infty,\,\neq 0}^{\infty} \quad,\quad \int_{-\infty}^{\infty} d\Omega \longrightarrow \sum_{N=-\infty,\,\neq 0}^{\infty} \quad,\quad \int_{-\infty}^{\infty} d\eta \longrightarrow \int_{\eta_i}^{\eta_f} d\eta\,,$$

and one of the equation in the completeness (18) is changed to the form $\int_{\eta_i}^{\eta_f} d\eta\, f_{F,\omega}^*(\eta)\, f_{F,\omega'}(\eta) = \eta_p\, \delta_{\omega,\omega'}$. Consequently, the junction coefficient $A_{\Omega\omega}^{(1)}$ can be calculated by the similar way to obtain (30),

$$A_{\Omega\omega}^{(1)} = \frac{e^{-i\omega r_0^\star}}{\eta_p} \sqrt{\left|\frac{\omega}{\Omega}\right|} \int_{\eta_i}^{\eta_f} d\eta\, e^{i\omega\eta - i\,\Omega[T(\eta) - R^\star(\eta)]}\,, \tag{46}$$

and $A_{\Omega\omega}^{(2)} = 0$ should also hold. Finally the Hawking spectrum (41) is modified to the following form,

$$I_H(\eta_i, \eta_f; \omega) = \sum_{N=1}^{\infty} \frac{1}{e^{2\pi\Omega/\kappa} - 1}\, D_H(\Omega, \omega)\,, \tag{47}$$

where ω and Ω are given by (45), and the deviation factor $D_H(\Omega, \omega)$ is given by (42).

Applying this finite time interval normalization to the simplest case of the previous subsection 4.2.1, the delta function in (43) is replaced by the Kronecker delta $\delta_{\omega,b_c\Omega}$. Then the spectrum (44) is normalized to the form (47), $I_H(\eta_i, \eta_f; \omega) = b_c(e^{2\pi\omega/b_c\kappa} - 1)^{-1}$.

4.3   Some models

In order to understand the properties of the normalized Hawking spectrum in the SCU (47), it is useful to consider some cases of the phase factor, $T(\eta) - R^\star(\eta)$, in the representation (46).

4.3.1   Case: the thermal spectrum

Here we look for the scale factor $a(\eta)$ which makes the spectrum (47) be the thermal one. According to the previous section 4.2, the phase factor, $T(\eta) - R^\star(\eta)$, should be linear of η,

$$T(\eta) - R^\star(\eta) = b\,\eta + p \quad,\quad b = \text{const} > 0 \quad,\quad p = \text{const}. \tag{48}$$

The spectrum (47) of this case becomes thermal, $I_H(\eta_i, \eta_f; \omega) = b(e^{2\pi\omega/b\kappa} - 1)^{-1}$. The differential of the phase factor (48) gives the following,

$$\frac{d\,a(\eta)}{d\eta} = \frac{\sqrt{1 - k\,r_0^2}}{r_0}\,a(\eta) - \frac{b}{r_0}\left(1 - \frac{R_g}{a(\eta)\,r_0}\right). \tag{49}$$

Though the simplest case, $a = $ constant, is the trivial solution of this equation, the general solution is a dynamical universe, $a \neq$ constant.

For sufficiently small b, we can set $da/d\eta|_{\eta_0} > 0$ at an initial time η_0, and generally the scale factor grows exponentially. If we set roughly $a(\eta) = e^{\alpha\eta}$, where α is a constant, then it is rewritten to the form $a(t) \propto t$ through the relation $dt = a(\eta)\,d\eta$. This is the asymptotic form of the scale factor for the open FRW universe at a remote future $t \to +\infty$. That is, one may think that the black hole in the open universe emits a thermal Hawking radiation. However the situation is not so simple, as discussed in the following two subsections.

4.3.2 Case: $T(\eta) - R^\star(\eta) \propto \eta^2$

In this subsection we treat a case that the scale factor gives the phase factor of the junction coefficient (46) as

$$T(\eta) - R^\star(\eta) = c\,\eta^2 + b\,\eta + p \quad,\quad c = \text{const} > 0 \quad,\quad b, p = \text{const.}$$

Taking the differential of this equation, we find

$$\frac{d\,a(\eta)}{d\eta} = \frac{\sqrt{1 - k\,r_0^2}}{r_0}\,a(\eta) - \frac{2\,c\,\eta + b}{r_0}\left(1 - \frac{R_g}{a(\eta)\,r_0}\right). \tag{50}$$

Generally for the positive initial value $a(\eta_0) > 0$ at an initial time $\eta_0 < -b/2c$, this equation makes the scale factor $a(\eta)$ grow exponentially. Therefore by the same discussion given in the previous case, the present case also includes the asymptotic form of the open FRW universe at a remote future. Though the rough estimates of the equations (49) and (50) gives the same conclusion that $a(t) \propto t$, but the difference of them is the detailed behavior of $a(t)$. For example, because $c > 0$, the velocity of the expansion $da/d\eta$ of the present case is smaller than that of the previous case for $\eta > 0$.

The spectrum (47) of the present case is expected not to be the thermal one, since the phase factor, $T(\eta) - R^\star(\eta)$, is different from that of the previous case. The normalized junction coefficient (46) is rewritten using the change of the conformal time, $\bar{\eta} = \sqrt{c\,\Omega}\,e^{i\pi/4}[\eta - (\omega - b\Omega)/2c\Omega]$,

$$A_{\Omega\omega}^{(1)} = \frac{e^{-ip_c}}{\eta_p\,\Omega}\sqrt{\frac{|\omega|}{c}}\int_{\bar{\eta}_i}^{\bar{\eta}_f} d\bar{\eta}\,e^{-\bar{\eta}^2}, \tag{51}$$

where we set $\Omega(= 2\pi N/T_p) > 0$, that is, $N = 1, 2, 3, \cdots$, and $p_c = \omega r_0^\star - i(\omega - b\Omega)^2/4c\Omega + p\Omega + \pi/4$. Because the integrand $e^{-\bar{\eta}^2}$ has no pole on the complex $\bar{\eta}$-plane, the integral in this representation of $A^{(1)}$ becomes

$$\int_{\bar{\eta}_i}^{\bar{\eta}_f} = -\int_{L_1} - \int_{L_2} - \int_{\Re(\bar{\eta}_f)}^{\Re(\bar{\eta}_i)}, \tag{52}$$

where the integration paths are shown at the figure 3. Note that the path L_2 is given by $\bar{\eta} = \Re(\bar{\eta}_i) + iy$ for $0 < y < \Im(\bar{\eta}_i)$ where $\Im$ denotes the imaginary part, then we find

$$\int_{L_2} d\bar{\eta}\,e^{-\bar{\eta}^2} = i e^{-\Re(\bar{\eta}_i)^2}\int_0^{\Im(\bar{\eta}_i)} dy\,e^{y^2 - 2i\Re(\bar{\eta}_i)y} \longrightarrow 0 \quad \text{as } |\eta_i| \to \infty,$$

where $\Re(\bar{\eta}_i)^2 = (c\,\Omega/2)\,[\eta_i - (\omega - b\,\Omega)/2c\,\Omega]^2$. Similarly it is found that $\int_{L_1} \to 0$ as $|\eta_i| \to \infty$. Therefore we obtain the approximate form of $A^{(1)}$ for sufficiently large η_i,

$$A_{\Omega\omega}^{(1)} \simeq \frac{e^{-ip_c}}{\eta_p\,\Omega}\sqrt{\frac{|\omega|}{c}}\int_{\Re(\bar{\eta}_i)}^{\Re(\bar{\eta}_f)} d\bar{\eta}\,e^{-\bar{\eta}^2} = \frac{e^{-ip_c}}{\eta_p\,\Omega}\sqrt{\frac{|\omega|}{c}}\,[\,\text{Erfc}(\Re(\bar{\eta}_i)) - \text{Erfc}(\Re(\bar{\eta}_f))\,],$$

where $\text{Erfc}(z) = \int_z^\infty du\,e^{-u^2}$ is the Gauss's error function. This gives the deviation factor (42) for large η_i,

$$D_H(\Omega, \omega) \simeq \frac{\omega}{c\,\Omega^2\eta_p^2}\left[\Delta E(\Omega, \omega)^2 + e^{2\pi\Omega/\kappa}\,\Delta E(\Omega, -\omega)^2\right.$$

$$\left. + 2e^{\pi\Omega/\kappa}\cos\left(\frac{b^2\Omega^2 + \omega^2}{4c\,\Omega} - 2p\,\Omega - \frac{\pi}{2}\right)\Delta E(\Omega, \omega)\,\Delta E(\Omega, -\omega)\right], \tag{53}$$
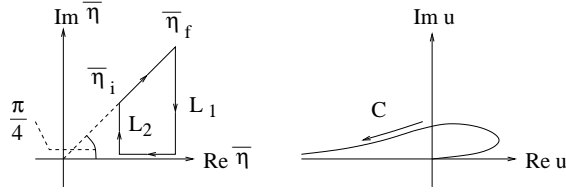
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Figure 3: The left figure shows the integration path for (52) in the $\bar{\eta}$-plane, and the right one for (56) in the u-plane.

where $\Delta E(\Omega, \omega) = \mathrm{Erfc}(\Re(\bar{\eta}_i)) - \mathrm{Erfc}(\Re(\bar{\eta}_f))$, $\Omega = 2\pi N/T_p$, $\omega = 2\pi n/\eta_p$, and $N, n = 1, 2, 3, \cdots$. Note that $\Re(\bar{\eta})$ depends on Ω and ω by definition of $\bar{\eta}$ given just before (51).

Here the following relation is important; $\mathrm{Erfc}(x) = (1/2\sqrt{x}) e^{-x^2/2} W_{-1/4, 1/4}(x^2)$, where $W_{\mu, \nu}(z)$ is the Whittaker function. Because the qualitative behavior of $W_{\mu, \nu}(z)$ is of dumping oscillation as a function of z, the normalized Hawking spectrum (47) of present case is expected to be oscillatory with decreasing amplitude as a function of ω. To estimate the high frequency amplitude, note that the asymptotic form of $\mathrm{Erfc}(z)$ is given using that of $W_{\mu, \nu}(z)$ as $\mathrm{Erfc}(z) \sim e^{-z^2}/2z$, as $|z| \to \infty$. Consequently we find $\Delta E(\Omega, \omega) \sim O(e^{-\omega^2/c\Omega} \sqrt{c\Omega}/\omega)$ as $\omega \to \infty$, then from (53), the high frequency amplitude of the Hawking spectrum can be roughly estimated to be $I_H(\eta_i, \eta_f; \omega) \sim O(e^{-2\omega^2}/\omega)$ as $\omega \to \infty$.

4.3.3 Case: $T(\eta) - R^\star(\eta) \propto \eta^3$

The third case is that the scale factor satisfies,

$$ T(\eta) - R^\star(\eta) = -c\,\eta^3 + b\,\eta + p \quad , \quad c = \mathrm{const} > 0 \quad , \quad b, p = \mathrm{const}. \tag{54} $$

Here note that even if there is a term proportional to η^2, it can be vanished by shifting the origin of η. The differential of this equation gives,

$$ \frac{d\,a(\eta)}{d\eta} = \frac{\sqrt{1 - k\,r_0^2}}{r_0} a(\eta) - \frac{-3\,c\,\eta^2 + b}{r_0} \left(1 - \frac{R_g}{a(\eta)\,r_0} \right). \tag{55} $$

From this equation we find that, if $a > 0$, the behavior of $a(\eta)$ is the exponential grow for $\eta > \sqrt{b/3c}$. That is, it can be said that this case also corresponds to the asymptotic form of the open FRW universe at a remote future, $a(t) \propto t$. As for the case of the previous subsection 4.3.2, the difference from the thermal case comes from the detailed behavior of $a(t)$. For example, because $c > 0$, the velocity of expansion $da/d\eta$ of the present case (55) should be larger than that of the thermal case (49) for $\eta > 0$.

The spectrum (47) of the present case is expected not to be the thermal one. To evaluate the normalized junction coefficient (46), we should carry out the integral, $\int_{\eta_i}^{\eta_f} d\eta \exp[ic\Omega\eta^3 + i(\omega - b\Omega)\eta]$. But it is difficult at least for the author to evaluate this integral. Therefore in order to guess roughly the behavior of the Hawking radiation, we try using the non-normalized representation (41). The non-normalized junction coefficient (30) is exactly calculated to be,

$$ A_{\Omega\omega}^{(1)} = e^{-i\omega r_0^\star - ip\Omega} \sqrt{\left| \frac{\omega}{\Omega} \right|} \frac{\sqrt{|\omega - b\Omega|}}{[3(c\Omega)^{1/3}]^{3/2}} $$
$$ \times \left[J_{-1/3} \left[2 \left(\frac{|\omega - b\Omega|}{3(c\Omega)^{1/3}} \right)^{3/2} \right] - \mathrm{sgn}(\omega - b\Omega) J_{1/3} \left[2 \left(\frac{|\omega - b\Omega|}{3(c\Omega)^{1/3}} \right)^{3/2} \right] \right], $$

where the integral representation of the Bessel function is used,

$$ \int_{-\infty}^{\infty} dx\, e^{i(qx)^3 + ikx} = \frac{2\pi\sqrt{|k|}}{(3q)^{3/2}} \left[J_{-1/3} \left[2 \left(\frac{|k|}{3q} \right)^{3/2} \right] - \mathrm{sgn}(k) J_{1/3} \left[2 \left(\frac{|k|}{3q} \right)^{3/2} \right] \right] \quad \text{for } q > 0. $$

Then the deviation factor D_H becomes,

$$ D_H(\Omega, \omega) = \frac{\omega}{27c\,\Omega^2} \left[|\omega - b\,\Omega|\,\Delta_+(\Omega, \omega)^2 + e^{2\pi\Omega/\kappa}\,(\omega + b\,\Omega)\,\Delta_-(\Omega, \omega)^2 \right. $$
$$ \left. + 2\,e^{\pi\Omega/\kappa}\,\sqrt{|\omega^2 - c^2\,\Omega^2|}\,\cos\left(2p\,\Omega\right)\,\Delta_+(\Omega, \omega)\,\Delta_-(\Omega, \omega) \right], $$

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where

$$\Delta_+(\Omega,\omega) = J_{-1/3}\left[2\left(\frac{|\omega-b\,\Omega|}{3(c\,\Omega)^{1/3}}\right)^{3/2}\right] - \text{sgn}(\omega-b\,\Omega)\,J_{1/3}\left[2\left(\frac{|\omega-b\,\Omega|}{3(c\,\Omega)^{1/3}}\right)^{3/2}\right]$$

$$\Delta_-(\Omega,\omega) = J_{-1/3}\left[2\left(\frac{\omega+b\,\Omega}{3(c\,\Omega)^{1/3}}\right)^{3/2}\right] + J_{1/3}\left[2\left(\frac{\omega+b\,\Omega}{3(c\,\Omega)^{1/3}}\right)^{3/2}\right].$$

This result denotes that the Hawking spectrum is oscillatory as a function of ω which is the frequency observed by a comoving observe at a remote future $\eta \to +\infty$. The amplitude can be roughly estimated to be $_W\langle 0|\,N_{F,\omega}\,|0\rangle_W \sim \omega^{1/2}$ as $\omega \to \infty$, where the asymptotic form of the Bessel function is used, $J_\nu(z) \sim \sqrt{2/\pi z}\,\cos[z - (2\nu+1)\pi/4]$ as $|z| \to \infty$. Because this result is not normalized, the high frequency spectrum diverges. But the qualitative behavior that the spectrum is oscillatory may hold even for the normalized one.

4.3.4 de-Sitter case: $a(\eta) = -c/\eta$

Here we turn our attention from the open universe to the other universe. Consider the de-Sitter case,

$$a(\eta) = \frac{-c}{\eta} \quad , \quad \text{for } \eta < 0 \text{ and } c = \text{const.} > 0.$$

The relation, $dt = a(\eta)d\eta$, gives $a(t) = \exp(t/c)$, that is, this is the inflationary universe.

From (10) and the definition of $R^\star(\eta)$, we obtain the phase factor,

$$T(\eta) - R^\star(\eta) = \frac{c\,r_0}{\eta} + \left(c\sqrt{1-k\,r_0{}^2} - R_g\right)\ln\left(\frac{1}{\eta} + \frac{R_g}{c\,r_0}\right) + p,$$

where p is the integral constant due to (10). Hereafter in this subsection, we assume that the junction surface Σ of the SCU is much larger than the black hole, $R_g \ll R(\eta) = -c\,r_0/\eta$. Then the approximation, $1/\eta + R_g/c\,r_0 \simeq 1/\eta$, becomes valid, and the normalized junction coefficient (46) is expressed as

$$A^{(1)}_{\Omega\,\omega} \simeq \frac{e^{-i\omega r_0^\star - ip\,\Omega}}{\eta_p}\sqrt{\left|\frac{\omega}{\Omega}\right|}\int_{\eta_i}^{\eta_f}d\eta\,e^{i(\omega\eta - c\,r_0\,\Omega/\eta)}\,\eta^{i\Omega\left(c\sqrt{1-k\,r_0{}^2}-R_g\right)}$$

$$= \frac{e^{-i\omega r_0^\star - ip\,\Omega}}{\eta_p}\sqrt{\left|\frac{\omega}{\Omega}\right|}\left(\frac{c\,r_0\,\Omega}{\omega}\right)^{-\nu}\int_{u_i}^{u_f}du\,e^{z/2\,(u-1/u)}\,u^{-1-\nu},$$

where $z = i\,2\sqrt{c\,r_0\,\Omega\,\omega}$, $\nu = -1 - i\Omega\left(c\sqrt{1-k\,r_0{}^2} - R_g\right)$, and the variable of integration is changed as $\eta \to u = \eta\sqrt{\omega/c\,r_0\Omega}$. Here let us take the approximations, $\eta_f \to 0$ and $\eta_i \to -\infty$, and further consider the modification, $\omega \to \lim_{\epsilon\to 0}\omega\,e^{-i\epsilon}$. Then the integral path can be analytically connected to the complex u-plane and deformed to C shown at the figure 3, which gives

$$A^{(1)}_{\Omega,\omega} \simeq \frac{-i\pi e^{i\omega r_0^\star - ip\,\Omega}}{\eta_p}\left(\frac{c\,r_0\,\Omega}{\omega}\right)^{-\nu/2}\sqrt{\left|\frac{\omega}{\Omega}\right|}\,H^{(1)}_\nu\left(i\,2\sqrt{c\,r_0\,\Omega\,\omega}\right), \tag{56}$$

where the integral representation of the Hankel function is used, $i\pi H^{(1)}_\nu(z) = -\int_C du\,e^{z/2\,(u-1/u)}\,u^{-1-\nu}$ for $\Re(z) > 0$. This is valid only for $\omega > 0$. But we can obtain the same result (56) for $\omega < 0$ by routine calculations with taking the branch cut of $\ln u$ in the upper half u-plane ($-1 = e^{-i\pi}$). Therefore the deviation factor is given as

$$D_H(\Omega,\omega) \simeq \frac{\pi^2 c\,r_0}{\eta_p{}^2}\left[\left|H^{(1)}_\nu\left(i\,2\sqrt{c\,r_0\,\Omega\,\omega}\right)\right|^2 + e^{2\pi\Omega/\kappa}\left|H^{(1)}_\nu\left(2\sqrt{c\,r_0\,\Omega\,\omega}\right)\right|^2\right.$$

$$\left. + 2\,e^{\pi\Omega/\kappa}\,\Re\left(e^{i\pi/2 - i2p\,\Omega}\left(\frac{c\,r_0\,\Omega}{\omega}\right)^{-1-\nu}H^{(1)}_\nu\left(i\,2\sqrt{c\,r_0\,\Omega\,\omega}\right)H^{(1)}_\nu\left(2\sqrt{c\,r_0\,\Omega\,\omega}\right)\right)\right].$$

This is essentially the dumping oscillation about ω.

The amplitude of the high frequency spectrum is estimated using the asymptotic form of the Hankel function, $H^{(1)}_\nu(z) \sim \sqrt{2/\pi z}\,e^{i(z-(2\nu+1)\pi/4)}$, to be, $H^{(1)}_\nu\left(i2\sqrt{c\,r_0\,\Omega\,\omega}\right) \sim O\left((\Omega\omega)^{-1/4}\times e^{-\sqrt{\Omega\omega}-\Omega}\right)$. Then the dominant term of D_H is the second term, and we find that $I_H(-\infty,0;\omega) \sim O(1/\sqrt{\omega})$ as $\omega \to \infty$.

4.3.5 Case: close to the thermal spectrum

Finally let us examine the case which is close to the thermal case. According to the subsection 4.3.1, the Hawking spectrum (47) should be reduced to the thermal one by a manipulation which makes the scale factor come to satisfy (49). For example, the slow cosmological expansion limit, $\dot{a}/a \to 0$, would reproduce the thermal spectrum. Therefore we consider the case,

$$T(\eta) - R^\star(\eta) = c\,\eta^l + b\,\eta + p \quad \text{as } c \to 0,$$

where l is an arbitrary natural number, and it is assumed that the scale factor is analytic as a function of η and consequently the phase factor $T - R^\star$ is also analytic. This manipulation, $c \to 0$, should include the limit $\dot{a}/a \to 0$.

As $c \to 0$, the junction coefficient (46) is calculated as,

$$
\begin{aligned}
A_{\Omega\,\omega}^{(1)} &\simeq \frac{e^{-i\omega r_0^\star - ip\,\Omega}}{\eta_p} \sqrt{\left|\frac{\omega}{\Omega}\right|} \int_{\eta_i}^{\eta_f} d\eta \left(1 - ic\,\Omega\,\eta^l\right) e^{i(\omega - b\,\Omega)\eta} \\
&= e^{-i\omega r_0^\star - ip\,\Omega} \sqrt{\left|\frac{\omega}{\Omega}\right|} \left[\delta_{\omega, b\Omega} - \frac{ic\,\Omega}{\eta_p} \int_{\eta_i}^{\eta_f} d\eta\,\eta^l\, e^{i(\omega - b\,\Omega)\eta} \right].
\end{aligned}
$$

Here by treating the Kronecker $\delta_{\omega, b\Omega}$ as a distribution and referring to the property of the delta function, $d\delta(x)/dx = -\delta(x)d/dx$, the integral in the second term becomes,

$$\int_{\eta_i}^{\eta_f} d\eta\,\eta^l\, e^{i(\omega - b\,\Omega)\eta} = \eta_p \left(\frac{1}{ib}\right)^l \delta_{\omega, b\Omega} \left(\frac{d}{d\Omega}\right)^l,$$

which gives

$$A_{\Omega\,\omega}^{(1)} \simeq e^{-i\omega r_0^\star - ip\,\Omega} \sqrt{\left|\frac{\omega}{\Omega}\right|}\, \delta_{\omega, b\Omega} \left[1 + (-i)^{l+1} \frac{c\,\Omega}{b^l} \left(\frac{d}{d\Omega}\right)^l \right],$$

where $\omega = 2\pi n/\eta_p$, $\Omega = 2\pi N/T_p$. This means $A_{\Omega\,-\omega}^{(1)} = 0$ for $\omega > 0$ and $\Omega > 0$, therefore we obtain

$$D_H(\Omega, \omega) = \left|A_{\Omega\,\omega}^{(1)}\right|^2 \quad \to \quad \frac{\omega}{\Omega}\, \delta_{\omega, b\Omega}^{\,2} \quad \text{as } c \to 0$$

$$\implies \quad I_H(\eta_i, \eta_f; \omega) \quad \to \quad \frac{b}{e^{2\pi\omega/b\kappa} - 1} \quad \text{as } c \to 0.$$

In the limit $c \to 0$, the second and third terms of the deviation factor D_H do not contribute to the Hawking spectrum $I_H(\eta_i, \eta_f; \omega)$, and the first term of D_H is reduced to that of the thermal case. However in the non-linear case, $c \neq$ so small, the second and third terms of D_H come to contribute to $I_H(\eta_i, \eta_f; \omega)$. That is, we can expect naively that the intensity of the non-thermal Hawking radiation is stronger than that of the thermal one due to the exponential factor $e^{\pi\Omega/\kappa}$ in D_H, where the mass of the black hole is the same for both of the cases.

5 Summary and discussion

5.1 Summary

How does a dynamical boundary condition like an expanding universe change the well-known properties of asymptotically flat black hole spacetimes? This paper is the first trial to treat this question, and deals with the effect of cosmological expansion on the Hawking radiation. The swiss cheese universe (SCU) has been adopted as a concrete model of background spacetime. With using the SCU we could avoid the issue how to define the black hole in an expanding universe, and the mass of the black hole could be defined by the Kumar mass computed on the spatial section of the junction surface Σ of the SCU. Further for simplicity, the SCU has been assumed to be two dimensional in order to neglect the curvature scattering of a massless scalar field of minimal coupling. It has also been assumed that the SCU was eternal, that is, no big bang and big crunch singularities would appear and the scale factor $a(\eta)$ could be defined in the infinite interval of the conformal time of the FRW-side, $-\infty < \eta < \infty$.

We have introduced a massless scalar field Φ of minimal coupling, and ignored the back reaction to the background SCU. The junction condition of Φ has required no potential confined on Σ. Therefore Φ was not reflected at Σ. Consequently, since we have neglected the curvature scattering, a mode function which is a monochromatic out-going one in the BH-side has been connected at Σ to a mode function in the FRW-side which is of a superposition of the out-going modes without including any in-going one, and so has been the in-going modes.

The scalar field has been quantized with the WX-mode, the TR-mode and the Cfl-mode. Because the event horizon bifurcated in the eternal SCU, we could make use of the discussion given for the asymptotically flat eternal Schwarzschild spacetime, and obtained the Bogoljubov transformation from the WX-mode to the TR-mode [9]. For finding the Bogoljubov transformation from the TR-mode to the Cfl-mode, it has been essential that the junction coefficients of the mode functions determined at Σ could be re-interpreted as the Bogoljubov coefficients. Finally by the successive Bogoljubov transformation from the WX-mode to the Cfl-mode via the TR-mode, we have obtained the Hawking radiation (41) as the particle of the Cfl-mode created on the vacuum state of the WX-mode. This Hawking radiation was measured by a comoving observer at a remote future $\eta \to +\infty$.

However, because the back reaction was ignored, the Hawking spectrum (41) has diverged as mentioned at the beginning of the section 4.2. This divergence arose from the infinite time interval of the observation. Therefore we have introduced the finite time interval normalization which restricted the observation time, $\eta_i < \eta < \eta_f$, and obtained the normalized Hawking spectrum (47).

5.2 Discussions

It is easily expected that the Bogoljubov transformation from the TR-mode to the Cfl-mode causes the difference of the resultant spectrum from the thermal one. Note that this Bogoljubov transformation does not include the cosmological particle creation, since the massless scalar field of minimal coupling in two dimensional spacetime is not scattered by the background curvature. Therefore the deviation factor D_H of the resultant spectrum (42) includes only the effect of the relative motion of the comoving observer to the TR-system. One may think that the resultant spectrum would be just a thermal one which would receive a red shift to a lower temperature due to the comoving motion. However it is obvious from (47) that the resultant Hawking spectrum is totally different from a thermal one, except the case of subsection 4.3.1. The reason why the spectrum is not generally a thermal one, is in the junction condition of Φ at Σ. If a mode function can be a monochromatic in both of the BH-side and the FRW-side, we can understand the junction condition as a simple Doppler effect. However as mentioned in the previous subsection, a monochromatic out-going mode in the BH-side goes through Σ into the FRW-side, then it turns to a superposition of out-going modes in the FRW-side. Therefore a monochromatic out-going mode received by a comoving observer in the FRW-side, should be a superposition of out-going modes in the BH-side when it would be traced back into the BH-side. This denotes that various frequencies Ω of the TR-mode are included in each monochromatic mode of frequency ω in the FRW-side. That is, so many Doppler effects take place at the same time in one Cfl-mode, which is represented by D_H. We interpret such the superposition of Doppler effects as the pure effect of the acceleration due to the cosmological expansion.

According to the subsection 4.3.1, it seems that the thermal Hawking spectrum is of a special case where the second and third terms in the representation (42) of D_H vanish, and that the Hawking spectrum (47) is generally different from the thermal one. However as examined in the subsection 4.3.5, a non-thermal spectrum is reduced to the thermal one by an appropriate manipulation, for example, by the slow cosmological expansion limit, $\dot{a}/a \to 0$. About the non-thermal Hawking spectrum, the subsections 4.3.2, 4.3.3 and 4.3.4 give the conclusion that the qualitative behavior of the non-thermal spectrum is the dumping oscillation as a function of the frequency ω measured by a comoving observer. Further as discussed at the end of the subsection 4.3.5, the intensity of the non-thermal Hawking radiation is stronger than that of the thermal one. Then we can describe a picture from the viewpoint of the black hole thermodynamics that *a black hole with an asymptotically flat boundary condition stays in a lowest energy thermal equilibrium state. When a black hole is put into a dynamical boundary condition, it is excited to a non-equilibrium state, and emits its mass energy with stronger intensity than the thermal one.* However, about our general aim given at the beginning of section 1, we could not search for a concrete example of the non-gravitational and non-equilibrium system which corresponds to a black hole in an expanding universe.

So far we have assumed two dimensional SCU. In extending to four dimension, there arise three problems due to the curvature scattering: (i) the initial vacuum state, (ii) the junction of Φ at Σ and (iii) the cosmological particle creation. For the first issue, as mentioned in the subsection 2.2.3, we should modify the vacuum state on which the particle creation is estimated. The second issue means that, even if $F = 0$ at (69), a reflection of Φ due to the curvature scattering takes place at Σ, that is, a monochromatic out-going mode in the BH-side should be a superposition of the out-going and in-going modes in the FRW-side. Both of these issues (i) and (ii) do not cause any change for our strategy explained in the section 2.2. However about the third issue (iii), we should add another step to our strategy. The new step is an extra Bogoljubov transformation from the Cfl-mode of initial time to that of the final time, which raises the cosmological particle creation. But such a Bogoljubov transformation will take a complicated form, and we can hardly estimate its form since it is difficult to solve the mode functions with considering the curvature scattering. However if one wants to research the effects of only the cosmological particle creation, it is the easiest model to consider the massive scalar field on two dimensional SCU [10].

Finally let us discuss about the assumption of the "eternal" SCU. As mentioned at the end of section 2.1, though the evolution of the scale factor is mathematically given by the equation (66), we dared to assume that the scale factor was a positive definite function $a(\eta) > 0$ for $-\infty < \eta < \infty$. With this assumption we could obtain the form of the junction

coefficients $A^{(1,2)}$ in the simple representation of (23). Then what should we modify if the equation (66) is taken into account? Because of the initial singularity $a(t=0)=0$, we should solve the junction conditions of the scalar field Φ for the interval $\eta_0 < \eta < \infty$, where η_0 is given by $a(\eta_0)\, r_0 = R_g$. This modifies the completeness (18) to

$$\int_{\eta_0}^{\infty} d\eta\, f_{F,\omega}^*(\eta)\, f_{F,\omega'}(\eta) = \pi\, \delta(\omega - \omega') + i e^{i(\omega'-\omega)\eta_0}\, \mathrm{pv}\, \frac{1}{\omega - \omega'}\,,$$

where pv is the Cauchy's principle value. The second term may raise some complicated term in the equation (23), which would make the following computations intricate. However if we can assume that

$$\int_{-\infty}^{\infty} d\omega'\, \mathrm{pv}\, \frac{A_{\Omega\omega'}^{(1,2)}}{\omega - \omega'}\, \phi_{F,\omega'}^{(1,2)} = 0\,, \tag{57}$$

then we can obtain the junction coefficients of the form (23) with modifying the interval of integration of I_A and J_A from $-\infty < \eta < \infty$ to $\eta_0 < \eta < \infty$. Further even if this modified junction coefficient makes the Hawking spectrum diverge, we can expect well that the same normalization method discussed in section 4.2 holds. Here, by considering the four dimensional case, we can find the validity of the assumption (57). In four dimensional case, we can also separate the variables on the FRW-side, $\Phi = f_F(\eta)\, h_F(r^\star)\, Y_{lm}(\theta, \varphi)$, where Y_{lm} is the spherical harmonics. It can be easily found that the portion $f_F(\eta)$ satisfies a Sturm-Liouville type ordinary differential equation. This implies that there is a complete orthogonal set $\{f_{F,\omega}\}$ satisfying (18) with appropriate range and measure about η and ω. Therefore it is not so bad to assume the background spacetime is the eternal SCU in order to extract the essence of the Hawking radiation in an expanding universe.

Acknowledgements

A   Four dimensional swiss cheese universe

The swiss cheese universe (SCU) [5] is the spacetime including a spherically symmetric black hole in an expanding universe, and obtained by connecting a Schwarzschild spacetime with a dust-dominated Friedmann-Robertson-Walker (FRW) spacetime at a given spherically symmetric timelike hypersurface, Σ, by the Israel junction condition with no energy density confined on the junction surface [6]. In this appendix, we show a sketch of constructing the SCU in four dimension. Hereafter as the terminology, let the word "BH-side" denote the spacetime region inside Σ where the metric is given by (58), and "FRW-side" for the region outside Σ where the metric is (59).

The metric of a Schwarzschild black hole is

$$ds_{BH}^2 = -\left(1 - \frac{R_g}{R}\right) dT^2 + \left(1 - \frac{R_g}{R}\right)^{-1} dR^2 + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,, \tag{58}$$

where $x_{BH}^\mu = (T, R, \theta, \varphi)$ is the Schwarzschild coordinate and $R_g = 2GM$, M is the mass of the black hole and G is the gravitational constant. The metric of a FRW universe is

$$ds_F^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]\,, \tag{59}$$

where $x_F^\mu = (t, r, \theta, \varphi)$ is the comoving coordinate, $a(t)$ is the scale factor and $k = \pm 1, 0$ is the spatial curvature. We can make the angular coordinates θ and φ to be common to both of these coordinates due to the spherical symmetry. Further we assume that the Friedmann equation, that is the evolution of $a(t)$, is of dust-dominated one,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\frac{\rho_0}{a^3}\,, \tag{60}$$

where $\rho_0\, (= \text{const.})$ is of free parameter at this moment, but will be fixed consistently in considering the junction condition later.

In constructing the SCU we choose Σ to be spherically symmetric and timelike. Therefore the angular coordinate θ and φ can be chosen as the spatial coordinates on Σ, and it is possible to use t as the temporal coordinate on Σ.

That is, the coordinate we use in Σ is expressed as $u^i = (t, \theta, \varphi)$. [5] The radial coordinates of Σ measured from BH-side and FRW-side are generally given as a function of t, like $R(t)$ and $r(t)$ respectively. However especially about $r(t)$, we can specify it to be constant, $r = r_0 = $ constant. To understand this feature of radial coordinate of Σ, it is important to notice that the mass of the black hole M is assumed to be constant. If $r(t) \neq$ constant, there should be an energy flow of dust-matter through Σ, since the dust-matter rests on the comoving coordinate. Consequently, because it is also assumed that no energy density is confined on Σ, the source of the energy flow should be the mass of black hole, that is, M should have time dependence. [6] This contradicts $M = $ constant. Hence $r(t) = r_0 = $ constant. On the other hand, another function $R(t)$ cannot be specified at this moment, but will be determined by the junction condition.

The Israel junction condition at the junction surface Σ with no energy density confined on Σ, is given by [6]

$$\mathbf{h}_{BH}|_\Sigma = \mathbf{h}_F|_\Sigma \tag{61}$$

$$\mathbf{K}_{BH}|_\Sigma = \mathbf{K}_F|_\Sigma, \tag{62}$$

where $\mathbf{h}_{BH}|_\Sigma$ and $\mathbf{K}_{BH}|_\Sigma$ are respectively the induced metric and the extrinsic curvature of Σ measured from the BH-side, and similarly to $\mathbf{h}_F|_\Sigma$ and $\mathbf{K}_F|_\Sigma$ with respect to the FRW-side. The components of $\mathbf{h}$ and $\mathbf{K}$ on Σ are respectively, $h_{ij} = g_{\mu\nu} x^\mu_{,i} x^\nu_{,j}$, and $K_{ij} = n_{\mu;\nu} x^\mu_{,i} x^\nu_{,j}$, where $x^\mu_{,i} = \partial x^\mu / \partial u^i$, $g_{\mu\nu}$ is the metric of four dimension, n^μ is the unit vector normal to Σ. In the FRW-side, the unit normal to Σ is obtained from the expression of Σ, $S_F \equiv r - r_0 = 0$,

$$n^\mu_F = \frac{S_F^{,\mu}}{\sqrt{|S_{F,\alpha} S_F^{,\alpha}|}} = \left(0, \frac{1}{a(t)} \sqrt{1 - kr_0^2}, 0, 0 \right). \tag{63}$$

In the BH-side, $S_{BH} \equiv R - R(t) = 0$ gives Σ. Using the the junction condition (61), we obtain the unit normal to Σ,

$$n^\mu_{BH} = \frac{S_{BH}^{,\mu}}{\sqrt{|S_{BH,\alpha} S_{BH}^{,\alpha}|}} = \left(\left[1 - \frac{R_g}{R(t)} \right]^{-1} \dot{R}(t), \left[1 - \frac{R_g}{R(t)} \right] \dot{T}(t), 0, 0 \right). \tag{64}$$

The results of the junction conditions (61) and (62), are summarized into three independent equations. One of them specifies the location of Σ in the BH-side, $R(t) = a(t) r_0$ in BH-side. The other two equations give the form of the function $T(t)$ and the evolution of the scale factor on Σ,

$$\frac{dT(t)}{dt} = \frac{\sqrt{1 - kr_0^2}}{1 - R_g/R(t)}, \tag{65}$$

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{Rg}{r_0^3} \frac{1}{a^3}, \tag{66}$$

Comparing the result (66) with the assumption (60), the consistent choice of the constant ρ_0 is

$$\rho_0 = \frac{M}{(4/3)\pi r_0^3}. \tag{67}$$

To draw a picture of the SCU, assume $R(t_c) > R_g$ holds at an arbitrary time t_c for the case that the FRW-side is open $k = -1$ or flat $k = 0$. Then, because $a(t)$ increases monotonically in these cases, the relation (67) provides us with the following picture of the SCU: in a dust-dominated FRW universe including no black hole, the dust-matter contained in a spherically symmetric region of comoving radius r_0 starts to collapse, and finally a spherically symmetric black hole of mass $M = (4/3)\pi r_0^3 \rho_0$ is produced in an expanding universe. The conformal diagram of this interpretation is shown at the figure 4, which makes it clear that a black hole can be defined in the SCU with open and flat FRW-side. On the other hand, it is impossible to define a black hole for the case of closed FRW-side. Because $a(t)$ decreases to zero in a finite future, no causal curve cannot avoid encountering the big crunch singularity, hence every spacetime point is included in the causal past of the singularity and we cannot define a black hole.

Finally as the definition of the mass of the black hole in the SCU, we adopt the Komar mass calculated on the junction surface Σ. The general form of the Komar mass Q_K [7] is

$$Q_K = \frac{1}{8\pi G} \oint_{\partial A} dS_{\mu\nu} \xi^{\mu;\nu}, $$

where A is a spacelike region, ∂A is the boundary of A, ξ is the timelike Killing vector in A and $dS_{\mu\nu}$ is the surface area form on ∂A. Using the Einstein equation and the Killing equation, it can be proved that Q_K is invariant under

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[5]It is also possible to use T as the temporal coordinate on Σ. The reason we chose t is just the later convenience.

[6]For more correct discussion, consider the Komar integral evaluated on an arbitrary closed two dimensional spatial surface including the spatial section of the event horizon. Then the resultant Komar mass is exactly M for the metric (58) as calculated at the end of this appendix A. This means that the energy flow through Σ affects directly the mass of the black hole, if Σ is located outside the event horizon.
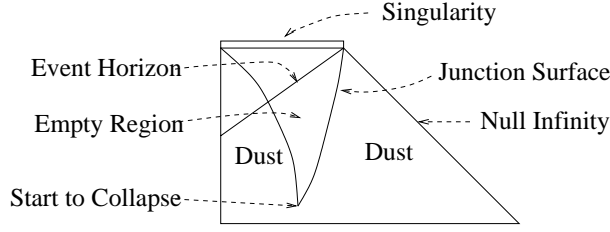
Figure 4: Conformal diagram of the SCU interpreted as the collapse of the dust-matter in flat or open FRW universe.

any deformation of A if ξ lasts to exist on the deformed region. For the SCU, ∂A is the spatial section of Σ at $T =$ constant, and we can let $\xi = \partial_T$, then the Komar mass is obtained to be $Q_K = M$. Here note that an important issue is left: the normalization of our Killing vector $\xi = \partial_T$ is not given at the junction surface nor at the spatial infinity of the FRW-side, but given at the asymptotically flat region of the full Schwarzschild spacetime which is not present in the SCU. Therefore we should say that, with accepting the vector ∂_T as a timelike Killing vector in the BH-side, the M appeared in (58) can be understood as a mass of the black hole in the SCU which is invariant under any deformation of the spatial hypersurface in the SCU. Further as a by-product of this normalization, the surface gravity κ of the event horizon is evaluated to be $\kappa = 1/2R_g$.

B Junction condition for scalar field

The junction condition for connecting a scalar field ϕ at a given hypersurface Σ is discussed in this appendix. As a terminology, we call the spacetime regions separated by Σ A-region and B-region, and denote a quantity Q evaluated in A-region and B-region as Q_A and Q_B respectively. We consider the four dimensional background spacetime in the Gaussian normal coordinate with respect to a hypersurface Σ, and the scalar field ϕ with an arbitrary potential $V(\phi)$. The metric is expressed as

$$ds^2 = \frac{1}{n^2} dx^{n\,2} + g_{ij}\, dx^i dx^j \,,$$

where x^i is the coordinate intrinsic to the hypersurface Σ, x^n is the coordinate vertical to Σ, and $n^\mu = g^{n\mu}$ is the unit normal vector to Σ, that is, $n^2 = 1$ if Σ is timelike or $n^2 = -1$ if Σ is spacelike. Hereafter we set the surface Σ is placed at $x^n = 0$. The Klein-Gordon equation is

$$\Box\phi - \frac{dV(\phi)}{d\phi} = 0 \,.$$

By integrating this equation along x^n direction through Σ, we find

$$\int_{-\epsilon}^{\epsilon} dx^n\, \Box\phi = \int_{-\epsilon}^{\epsilon} dx^n\, \frac{dV}{d\phi} \,.$$

On the other hand by definition of the d'Alembertian, we obtain

$$\int_{-\epsilon}^{\epsilon} dx^n\, \Box\phi = \int_{-\epsilon}^{\epsilon} dx^n\, \partial_n \partial^n \phi + \int_{-\epsilon}^{\epsilon} dx^n\, \left(\Gamma^n_{n\mu}\partial^\mu\phi + \nabla_i\nabla^i\phi\right) \longrightarrow \partial^n\phi_A - \partial^n\phi_B \quad \text{as } \epsilon \to 0 \,,$$

where $\partial^n = n^\mu\partial_\mu\phi$, and it is assumed that the A-region is the region of $x^n > 0$ and B-region is of $x^n < 0$. Here assume that there is a potential confined on the surface Σ such as

$$\frac{dV(\phi)}{d\phi} = F\delta(x^n) \,,$$

where F denotes a "surface potential force" derived from the potential confined on Σ. Further it is natural to require the continuity of ϕ on Σ. Hence finally we obtain the junction condition of a scalar field on a given surface Σ to be

$$\phi_A|_\Sigma = \phi_B|_\Sigma \tag{68}$$

$$n^\mu_A\partial_\mu\phi_A|_\Sigma - n^\mu_B\partial_\mu\phi_B|_\Sigma = F \,, \tag{69}$$

If no potential is confined on the surface Σ, the surface force vanishes $F = 0$. This junction condition of a scalar field is very similar to the Israel junction condition for connecting geometries of spacetimes [6] in the sense that a kind of source of the field under consideration is required in order to satisfy the field equation even on the junction surface.

C   Second term of equation (27)

In this appendix we discuss how the second term of (27) vanishes and what it means. To begin with, we need to know the behavior of the phase $-i\,\Omega[T(\eta) - R^\star(\eta)]$. Because the background is the eternal SCU, the junction surface Σ reaches the past temporal infinity as $\eta \to -\infty$ and the future temporal infinity as $\eta \to +\infty$. Then the equations (3) give $T(\eta) - R^\star(\eta) \to \pm\infty$ as $\eta \to \pm\infty$.

Note that the mode functions $\psi^{(\pm)}_{BH,\Omega}$ should be the basis of the general solution of the Klein-Gordon equation,

$$\Phi = \int_{-\infty}^{\infty} d\Omega \left[E_\Omega^{(+)} \, \psi_{BH,\Omega}^{(+)} + E_\Omega^{(-)} \, \psi_{BH,\Omega}^{(-)} \right] ,$$

where the coefficients $E^{(\pm)}$ depend only on Ω. This means that we should consider the property of the modes $\psi_{BH,\Omega}^{(\pm)}$ under the integration with a function of Ω. Let $H(\Omega)$ be a test function, we can calculate as [7]

$$\int_{-\infty}^{\infty} d\Omega \, \frac{H(\Omega)}{\sqrt{|\Omega|}} \, e^{-i\,\Omega[T(\eta)-R^\star(\eta)]}$$

$$= \frac{i}{T(\eta)-R^\star(\eta)} \frac{H(\Omega)}{\sqrt{|\Omega|}} \, e^{-i\,\Omega[T(\eta)-R^\star(\eta)]} \Bigg|_{\Omega=-\infty}^{\Omega=+\infty} - \int_{-\infty}^{\infty} d\Omega \, \frac{i}{T(\eta)-R^\star(\eta)} \left(\frac{d}{d\Omega} \frac{H(\Omega)}{\sqrt{|\Omega|}} \right) e^{-i\,\Omega[T(\eta)-R^\star(\eta)]} ,$$

then because $T(\eta) - R^\star(\eta) \to \pm\infty$ as $\eta \to \pm\infty$,

$$\int_{-\infty}^{\infty} d\Omega \, \frac{H(\Omega)}{\sqrt{|\Omega|}} \, e^{-i\,\Omega[T(\eta)-R^\star(\eta)]} = 0 \quad \text{as } \eta \to \pm\infty .$$

Hence the second term of (27) vanishes [8] whenever the mode functions $\psi_{BH,\Omega}^{(\pm)}$ are considered under the integration with respect to Ω,

$$\frac{\mathrm{sgn}(\bar\eta)\,\pi}{(4\pi)^{3/2}\sqrt{|\Omega|}} \; e^{-i\,\Omega[T(\bar\eta)-R^\star(\bar\eta)]} \Bigg|_{\bar\eta=-\infty}^{\bar\eta=+\infty} = 0 . \tag{70}$$

Further from this convergence, the equations (26) denote $I_A(\Omega,\omega) = -J_A(\Omega,\omega)$ whenever the mode functions $\psi_{BH,\Omega}^{(\pm)}$ are considered under the integration with respect to Ω.

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[7] Mathematically this is the Riemann-Lebesgue Integral Theorem.
[8] In an exact word, this is the weak convergence.